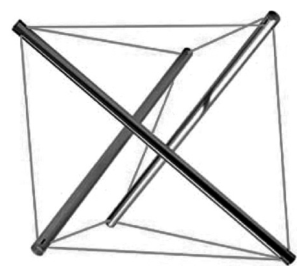
WHAT IS... a Tensegrity?

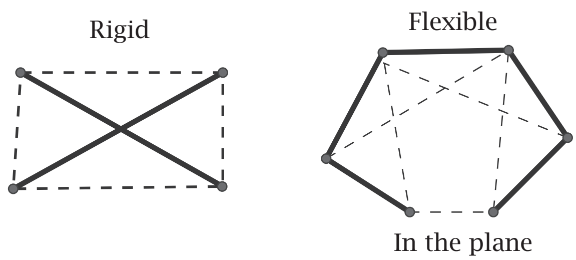
Robert Connelly

In the late 1940s, a young artist named Kenneth Snelson showed some of his string-and-stick sculptures to R. Buckminster Fuller. Out of this interaction the term "tensegrity" was born. These sculptures were quite surprising. The sticks were suspended in midair and supported by thin wires that were almost invisible. Fuller chose the word tensegrity to describe such structures because of their *tensional integrity*, and it has stuck throughout the years. The following shows a picture of one of the first small tensegrities that Snelson made with just three sticks (taken from his Web page). Since then, Snelson has made numerous other tensegrity sculptures, many quite large, and they are on display throughout the world.



But why do these structures hold up? Why are they rigid? What does it mean to be rigid? How can we model them mathematically?

The natural model is to define a tensegrity as a finite graph G whose vertices $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$, the *configuration*, are points in some Euclidean space $\mathbb{E}^d$ with two types of edges labeled *cables*, corresponding to the strings, and *struts*, corresponding to the sticks. The whole tensegrity is denoted as $G(\mathbf{p})$. The cables are constrained not to get longer, while the struts are constrained not to get shorter. A tensegrity $G(\mathbf{p})$ is defined to be *rigid* if, for any other configuration $\mathbf{q}$ in $\mathbb{E}^d$ sufficiently close to the configuration $\mathbf{p}$ and satisfying the cable and struts constraints of $G(\mathbf{p})$, $\mathbf{q}$ is rigidly congruent to $\mathbf{p}$. Some examples are below. Note that cables and struts can cross with no effect. A *flexible* tensegrity is one that is not rigid, and it necessarily has a smooth motion that is not a rigid motion of the whole tensegrity.



Struts are shown as solid line segments and cables as dashed line segments.

Static Rigidity

There are two principal methods to show rigidity of a tensegrity. Both methods involve the notion of a *stress* in the structure, which, for mathematical purposes, is just a scalar $\omega_{ij} = \omega_{ji}$ associated to every cable or strut connecting the ith vertex $\mathbf{p}_i$ to the jth vertex $\mathbf{p}_j$. We set $\omega_{ji} = 0$ when $\{i, j\}$ is not an edge of G. A stress is *proper* if $\omega_{ij} \geq 0$ for a cable and $\omega_{ij} \leq 0$ for a strut. The first method is derived from the linearization of the distance constraints. At each vertex $\mathbf{p}_i$ consider a force vector $\mathbf{F}_i$, the *equilibrium load*, such that these forces do not have any linear or angular momentum on the configuration. This means, summing over all i, that

$$\sum_i \mathbf{F}_i = 0 \quad \text{and} \quad \sum_i \mathbf{F}_i \wedge \mathbf{p}_i = 0.$$

In dimension 2 or 3 the wedge product can be replaced by the usual cross product. A tensegrity $G(\mathbf{p})$ is *statically* rigid if every equilibrium load $\mathbf{F} = (\mathbf{F}_1, \ldots, \mathbf{F}_n)$ can be resolved by a proper stress

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$\omega = (\ldots, \omega_{ij}, \ldots)$ in the sense that, for each i, $\omega_{ij} = \omega_{ji}$ and

$$\mathbf{F}_i + \sum_j \omega_{ij}(\mathbf{p}_j - \mathbf{p}_i) = 0.$$

This condition is then equivalent to there being a solution to a system of linear equations with linear inequality constraints on the stress ω, namely, a linear programming feasibility problem. In particular, if there are too few cables and struts, there will always be an equilibrium force that cannot be resolved by a proper stress. Suppose a tensegrity with n vertices and m cables and struts is statically rigid in dimension d. There are dn coordinate variables, and $d(d+1)/2$ is the dimension of the equilibrium loads (assuming the configuration does not lie in a lower-dimensional subspace). Then $m \geq dn - d(d+1)/2 + 1$. (The $+1$ arises due to inequality constraints.) In dimension 2, $m \geq 2n - 2$, and in dimension 3, $m \geq 3n - 5$. It is a basic result that, for a tensegrity, static rigidity is sufficient but not necessary for rigidity.

Prestress Stability

For example, the Snelson tensegrity in the first figure has $n = 6$ and $m = 12 < 3n - 5$, so it is not statically rigid. Nevertheless, it is rigid. Indeed, many of the tensegrities constructed by artists are statically underbraced. Although, from an engineering perspective, nonstatically rigid structures are often considered too soft to be dependable, we still want to detect and analyze such tensegrities. Basically, we can ensure that the tensegrity is rigid if the configuration minimizes an underlying energy function, which depends only on the lengths of the cables and struts, and if the minimizing configuration is unique up to rigid motions of the whole space. For example, static rigidity can be regarded as achieving a local minimum for reasonable choices of an energy function. Although the Snelson tensegrity and others like it have external loads that cannot be resolved at the given configuration, the tensegrity deforms slightly to resolve them. Indeed, even a statically rigid structure will deform under any nonzero load. When an energy function at the given configuration is at a local minimum due to the second derivative test, as in analysis, then the tensegrity is said to be *prestress stable*.

A *self-stress* for a tensegrity is a stress that resolves the zero load. Suppose that ω is a proper self stress for a tensegrity $G(\mathbf{p})$. Fix ω. Define a quadratic form $E(\mathbf{p})$ on the space of all configurations $\mathbf{p}$ in $\mathbb{E}^d$ by

$$E(\mathbf{p}) = \sum_{i<j} \omega_{ij} |\mathbf{p}_i - \mathbf{p}_j|^2.$$

It turns out that the matrix of E is the Kronecker product of d copies of an n-by-n matrix Ω with identity, where n is the number of vertices of the graph G, since the energy decouples into the value on each coordinate separately. For $i \neq j$, the i,jth entry of Ω is $-\omega_{ij}$, while the diagonal entries are such that the row and column sums of Ω are 0.

The stress energy E and the associated stress matrix Ω have some interesting properties.

1) If ω is a proper self-stress for the tensegrity $G(\mathbf{p})$ and the vertices of $\mathbf{p}$ span a d-dimensional (affine) linear subspace of $\mathbb{E}^d$, then the kernel of the associated stress matrix Ω is at least $(d+1)$-dimensional.
2) If ω is a proper self-stress for the tensegrity $G(\mathbf{p})$ in $\mathbb{E}^d$, the kernel of the associated stress matrix Ω is $(d+1)$-dimensional, and $G(\mathbf{q})$ is another tensegrity for another configuration $\mathbf{q}$ with the same self stress ω, then $\mathbf{q}$ is an affine image of $\mathbf{p}$.
3) If ω is a proper self-stress for the tensegrity $G(\mathbf{p})$ in $\mathbb{E}^d$, the kernel of the associated stress matrix Ω is $(d+1)$-dimensional, Ω is positive semidefinite, and $G(\mathbf{q})$ is a tensegrity for a configuration $\mathbf{q}$ with cables no longer and struts no shorter, then $\mathbf{q}$ is an affine image of $\mathbf{p}$.

Note that Property 3 does not quite say that the tensegrity is rigid, just that it is rigid up to affine motions, and they must preserve the lengths of the edges $\{i,j\}$, where $\omega_{ij} \neq 0$. Nevertheless, this is close enough in many circumstances. For a tensegrity $G(\mathbf{p})$ with a self-stress ω, its *stressed edge directions* are lines through the origin determined by the vectors $\mathbf{p}_i - \mathbf{p}_j$, where $\omega_{ij} \neq 0$. Think of these lines as points in the projective space $\mathbb{RP}^{d-1}$ of dimension $d-1$. We say that the stressed edge directions of a tensegrity $G(\mathbf{p})$ with self-stress ω *lie on a conic at infinity* if, as points in $\mathbb{RP}^{d-1}$, they lie on a conic. Then it is a pleasant exercise to show that, if the stressed edge directions of a tensegrity $G(\mathbf{p})$ do not lie on a conic at infinity, then there is no affine map of the configuration $\mathbf{p}$ that satisfies the cable and strut constraints on the stressed edges other than a rigid motion of the whole space $\mathbb{E}^d$, a congruence.

Putting this all together, we get the following basic result.

Theorem 1. *Suppose that $G(\mathbf{p})$ is a tensegrity in $\mathbb{E}^d$ with n vertices and a proper self-stress ω with associated stress matrix Ω, and the following hold:*

1) *The rank of Ω is $n-d-1$.*
2) *Ω is positive semidefinite.*
3) *The stressed directions of $G(\mathbf{p})$ do not lie on a conic at infinity.*

If $\mathbf{q}$ is another configuration in any $\mathbb{E}^D \supset \mathbb{E}^d$ satisfying the cable and strut constraints of $G(\mathbf{p})$, then $\mathbf{q}$ is congruent to $\mathbf{p}$.

I call any tensegrity that satisfies the three conditions of Theorem 1 *superstable*, and this sort of stability is a strong example of a structure being prestress stable. Quite a few of the tensegrities of Snelson and other artists are superstable. From a geometric point of view, the very strong conclusion of Theorem 1 about global reconfigurations is also interesting.

Global Rigidity

An edge of a tensegrity can also be regarded as both a cable and a strut, in which case it is called a *bar*; and if all the edges are bars, it is called a *bar framework*. In other words, bars are not permitted to change their lengths at all. From an engineering perspective, a tensegrity is just a bar framework where some bars can support only tension—these are the cables—and others can support only compression—these are the struts. A bar framework, or tensegrity, $G(\mathbf{p})$ is called *globally rigid* in $\mathbb{E}^d$ if any other configuration $\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_n)$ in $\mathbb{E}^d$ that satisfies the constraints of $G(\mathbf{p})$ is congruent to $\mathbf{p}$. While Theorem 1 gives conditions for a tensegrity to be globally rigid in all dimensions, the following result (see [3]) is for bar frameworks in a fixed $\mathbb{E}^d$. A configuration $\mathbf{p}$ in $\mathbb{E}^d$ is *generic*, which means it is typical, if the coordinates of all of the points do not satisfy any nonzero polynomial with integer coefficients.

Theorem 2. *A bar framework $G(\mathbf{p})$ with $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_n)$ generic is globally rigid in $\mathbb{E}^d$ if and only if G is the complete graph (all vertices connected to all others) on $d+1$ or fewer vertices or it has a self-stress ω and corresponding stress matrix Ω of rank $n-d-1$.*

There has been a lot of activity applying the ideas here to packing, granular materials, and point location in computational geometry, for example. One instance is that packings of spherical disks in a polyhedral container can be regarded as tensegrities with all struts connecting centers of touching disks to each other and the boundary of the container. Some of the results described here can be found in [1], [2], [3].

References

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