A Gallery of Algebraic Surfaces

Bruce Hunt

January 8, 2001

Introduction

The notion of a surface is a very classical one in technology, art and the natural sciences. Just to name a few examples, the roof of a building, the body of string instrument and the front of a wave are all, at least in idealized form, surfaces. In mathematics their use is very old and very well developed. A very special class of (mathematical surfaces), given by particularly nice equations, are the *algebraic surfaces*, the topic of this lecture. With modern software, one can make beautiful images of algebraic surfaces, which allow us to visualize important mathematical notions; explaining this is the object of this talk.

Several years ago I started to make pictures of algebraic surfaces. Originally I was involved in an open-day at the University of Kaiserslautern, and the pictures were an attempt to motivate youngsters in getting interested in mathematics. Later I made a gallery of these pictures for the Web, which is located at:

`http://www.mathematik.uni-kl.de/~wwwagag/Galerie.html`.

For my talk, which I was invited to give by Claude Bruter who had seen the gallery, I again took up to making some new images, so that the presentation here contains many images which are not presently available on the net. In addition, several colleagues and I created in Kaiserslautern some movies, which I showed at Maubeuge, some frames of which are also presented here.

I have attempted to show how these computer images can be used to visualize non-trivial mathematical concepts. After a brief introduction to the notion (definition) of algebraic surface in the first section, I consider successively three mathematical aspects to which the pictures yield a vivid visualization. The first is the notion of *symmetry*, certainly one of the most profound and important ones in all of mathematics. Often times a mathematical problem can only be treated for objects with a sufficient amount of symmetry. Of course, mathematicians have an abstract understanding of what symmetry is, but the pictures enable non-mathematicians to "see" it (mathematical symmetry, more than just the naive notion in our everyday vocabulary). Next we consider singularities of algebraic surfaces. Quite generally in mathematics, singularities are often of central interest, creating also most of the problems in the consideration of examples, as the "general theory" does not hold at these points. With the images of the algebraic surfaces, one can "see" what these singularities look like. Again, mathematicians have a kind of abstract understanding of singularities, but often do not really know what they look like, so real pictures of them are quite exciting. As a final topic, one rather specific to algebraic geometry (as opposed to differential geometry or topology), we consider some easy problems of *enumerative geometry*. In particular, with the computer images one can see the 27 lines on a smooth cubic surface.

Acknowledgements: I have made all images presented here and in the gallery with public domain software, a ray tracing system called VORT (Very Ordinary Ray Tracing), available via anonymous FTP

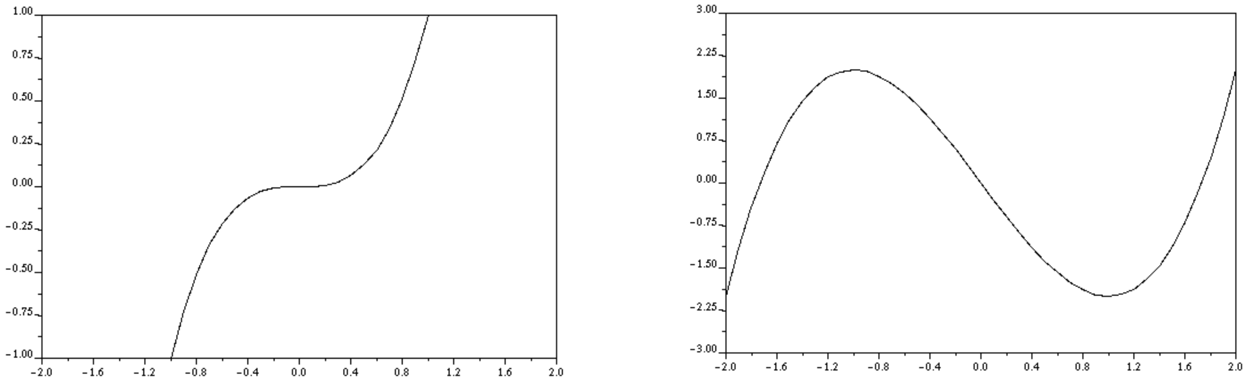


Figure 2: The graphs of the functions $f(x) = x^3$ and $f(x) = x^3 - 3x$

from `ftp://gondwana.ecr.mu.oz.au/pub/vort.tar.gz`. For the introduction to this system and help with creating the movies, it is my pleasure to thank Rüdiger Stobbe (at that time also at the University of Kaiserslautern). More help with creating movies was provided by Christoff Lossen, and in addition help with equations was provided by Duco von Straten (now University Mainz). The present system administrator at Kaiserslautern, David Ilsen, as well as his predecessor Hans Schönemann, were very helpful in getting VORT installed on my present computer and getting the equations I used in Kaiserslautern to me. My thanks to them all. Finally, my present employer, a state bank in Germany, was gracious enough to allow me to visit Maubeuge as a business trip. For that and more support, I thank them also.

1 Algebraic surfaces

To start this talk, I would like to motivate the notion of algebraic surface for non-mathematicians. For this, recall from your school courses the *graph of a function.* Typically one starts with the function $f(x) = x^2$, the graph of which is plotted in the (x, y)-plane by setting $f(x) = y$, i.e., $y = x^2$. You may remember that the graph of this function is a parabola, see Figure 1. The special point of the graph $(x, y) = (0, 0)$ is the *minimum* of this function. As a next example, consider $f(x) = x^3$, i.e., the graph given in the (x, y)-plane by $y = x^3$. You might recall that this graph is as in Figure 2, and here the special point at $(0, 0)$ is now not a minimum or maximum, but an *inflection point.* Finally, a more typical example is given by taking the slightly different function $f(x) = x^3 - 3x$ which now has a local minimum at $x = 1$ and a local maximum at $x = -1$.[1] Now, for any such graph, given by an equation $f(x) = y$ as above, we can form the *equation for a curve as a zero locus:*

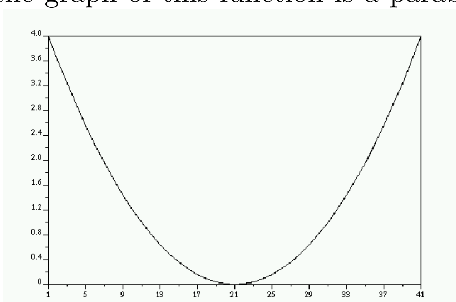


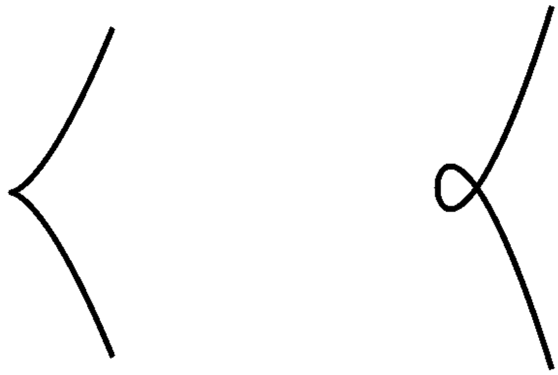
Figure 1: The graph of the function $f(x) = x^2$

$$g(x, y) := y - f(x) = 0. \tag{1}$$

The function g of two variables is defined by the expression (1). Thus, the *graph of a function* is a *special case* of a *curve* in the plane. If, moreover, $f(x)$ is a polynomial in x (a sum of *powers of* x) (as opposed to more complicated functions of x like sin, cos or tan) with certain coefficients, then the graph is said to be *algebraic*, and again, more generally, a curve defined by an equation (1) is said to be an *algebraic curve*, if $g(x, y)$ is a polynomial in two variables, i.e., $g(x, y) = \sum a_{ij} x^i y^j$.

Perhaps the simplest example of a curve which is not a graph is the so-called *Niel parabola* or *cuspidal*

[1] This is obtained by considering the derivatives of f, $f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x + 1)(x - 1)$, $\quad f''(x) = 6x$, hence $f''(1) = 6 > 0$ and $f''(-1) = -6 < 0$.



(a) The cuspidal cubic curve $x^3 - y^2 = 0$

(b) The cubic curve with a double point $x^3 - x^2 - y^2 = 0$

Figure 3: Two singular curves which are not graphs of functions

cubic, given by the equation $g(x,y) = x^3 - y^2 = 0$, which is depicted in Figure 3. Note that the point (0,0) for this curve is even more important than for the functions above: here we have an example of an algebraic curve with a *singular point* at (0,0). Another example of this kind is given by the *cubic with a double point*, given by the equation $x^3 - x^2 - y^2 = 0$, depicted in the second picture in Figure 3. There are two branches at $(0,0)$, determined by the two *tangents* which one gets by taking the derivatives, $x = 0$ and $x = 1$.

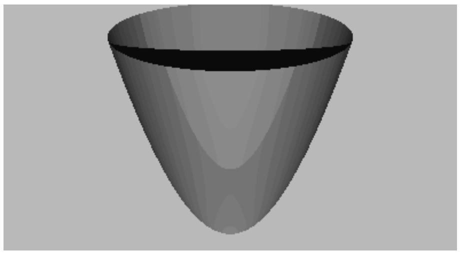


Figure 4: The graph of the function $g(x,y) = x^2 + y^2$

Now a statement that mathematicians love is: "What we can do once we can do again." So we now start with the function $g(x,y)$ and consider *its* graph, that is $z = g(x,y)$ in three-space. Plotting the graph now results in a landscape (which is referred to in mathematics as a *surface*). For $g(x,y) = x^2 + y^2$, we get the three-dimensional analog of the first example above, see Figure 4. This surface has a special property: it has a rotational *symmetry*, i.e., it is like a top which can spin on its vertex at $(0,0,0)$. A more interesting example is obtained by starting with our function $g(x,y) = x^2 - y^3$, which we explained above is the simplest example of a curve which is not a graph. In this case we get a surface $x^2 - y^3 - z = 0$, which is pictured in Figure 5. As opposed to the previous surface which is a *quadric*, i.e., its equation has degree 2, this is an example of a surface of degree 3, a cubic.

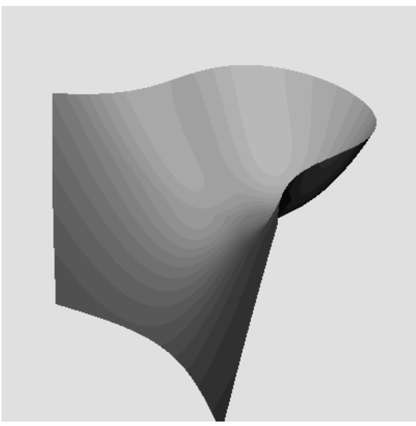


Figure 5: The algebraic surface with the equation $x^2 - y^3 - z = 0$.

At any rate, we do again as we did in the discussion of curves and consider the surfaces given by equations $\{h(x,y,z) = 0\}$, with the case of a graph being characterized by the condition $h(x,y,z) = z - g(x,y)$. This surface is said to be *algebraic*, if $h(x,y,z)$ is a polynomial in x, y, z: $h(x,y,z) = \sum a_{ijk} x^i y^j z^k$. Again, the simplest example of h which is not the graph of a function is $\{x^2 + y^2 - z^2 = 0\}$, shown in Figure 6. Note that this surface has a *singular point* at $(x,y,z) = (0,0,0)$.

For the mathematicians in the audience which I have been boring up to now let me briefly explain in more detail what an algebraic surface is. What we have been considering above belongs in the realm of *real algebraic geometry*, which is not the beautiful theory studied by algebraic geometers. Instead of working over the reals $\mathbb{R}$, classical algebraic geometry works over the complex numbers $\mathbb{C}$, and an algebraic surface is given by an equation $\{(x,y,z) \in \mathbb{C} | h(x,y,z) = 0\}$ for a polynomial h with complex coefficients. Consequently, this object is actually a four-dimensional space (naturally embedded in a six-dimensional one). Hence, viewing an algebraic surface as a topological space, it is four-dimensional topology which is relevant. Furthermore, what we have done above is *affine* geometry, and what is more interesting (in the wonderful theorems one gets) is *projective* geometry, so that an algebraic surface is a *compact* four-dimensional topological space, and the affine pictures above are real slices, locally around some point on the surface. The affine surface is the complement of a "curve at infinity" on the projective surface.

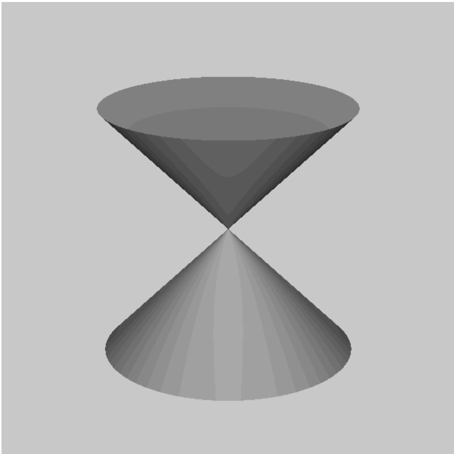


Figure 6: The surface $x^2+y^2-z^2=0$, which is not the graph of a function.

For the non-mathematicians which might be surprised that starting with something compact (which can be held in your hand) and taking away some curve, one gets something which is infinite, consider the following. If you take a sphere in your hand and delete the north pole, then a so-called *stereographic projection* maps it to the plane, which is infinite. This projection maps a point P (see Figure 7) to the point Q in the (x,y)-plane which is where the ray $\vec{OP}$, emanating at the north pole O and passing through P, intersects the plane.

The passage to projective geometry is affected by *homogenizing* the polynomial h, which is done by setting:

$$x=\frac{x_1}{x_0},\quad y=\frac{x_2}{x_0},\quad z=\frac{x_3}{x_0},$$

then multiplying $g\left(\frac{x_1}{x_0},\frac{x_2}{x_0},\frac{x_3}{x_0}\right)$ by the smallest power of x_0 necessary to clear denominators. The result is a homogenous polynomial $g_{hom}(x_0,x_1,x_2,x_3)$, and the projective surface is

$$\{g_{hom}(x_0,x_1,x_2,x_3)=0\}\subset\mathbb{P}^3,$$

where $\mathbb{P}^3$ denotes the three-dimensional projective space over $\mathbb{C}$ with homogenous coordinates (x_0,x_1,x_2,x_3).

Many properties of algebraic surfaces hold only for projective surfaces over the complex numbers. For example:

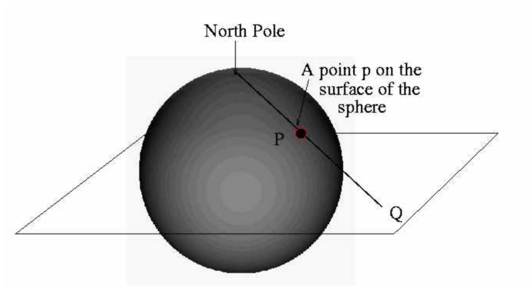


Figure 7: The stereographic projection

- A generic line in $\mathbb{P}^3$ meets an algebraic surface S in a fixed number d of points on the surface. This number d is called the *degree* of the surface. We have met examples of degrees 2 and 3 above.
- Any two smooth quadric surface are isomorphic (complex analytically)and have the same symmetry group, which is $SO(4)\cong SO(3)\times SO(3)$. Such a quadric surface is ruled; more precisely it is isomorphic to $\mathbb{P}^1\times\mathbb{P}^1$, where $SO(3)$ is the symmetry group of $\mathbb{P}^1$.
- Any two smooth surfaces of the same degree are *diffeomorphic* to one another, thus in particular have the same topological type. There is a finite number of *moduli* of such surfaces, i.e., parameters giving rise to complex analytically distinct surfaces.
- The notion of *duality* holds in projective space. This states that to every point there is a unique plane associated with it (the dual), and conversely, to every plane there is a unique point associated with it. An example of an application of this notion, the dual variety, is considered below.

2 Symmetry

A short dialog between a mathematician and an artist. The mathematician tries to explain to him the meaning of some mathematical concepts, first that of symmetry.

Artist: "I have often hear that mathematicians have a keen sense of esthetics. Can you explain this to me?"
Mathematician: "As a first point I'll try to elucidate the notion of symmetry. Look at this ball I have in my hand. In your opinion, is it symmetric?"
Artist: "It's the most symmetrical object there is. It has a certain symbolic importance because of that."
Mathematician: "Is it only the shape, or do you associate with it some other property which gives rise to a characterization of 'ball'?"
Artist: (After thinking a moment) "The shading, the coloring gives a unique pattern."
Mathematician: "Now let me describe another property, which we mathematicians see in this object. Instead of the shape of the ball, consider the fact that it *roles*. Phrased differently, we consider the symmetry of the object as its set of motions; in the case of the sphere these are the rotations you get by rolling the ball around on the table. Note that this is different than, for example, the case of an egg, which wobbles – it has less symmetry."

The "Erlanger Programm", put forth by Felix Klein in 1872, is the basis for a modern mathematical treatment of symmetry. It states that a property should be considered as part of some geometry, if it is *invariant* under the set of all automorphisms (motions) of the geometry. We have seen this above in the case of *continuous* groups of symmetry. However, we are actually more interested in *discrete* groups of symmetry, and even more specifically, in *finite* symmetries, i.e., objects which are invariant under finitely many motions of some kind. Think of, instead of a ball, a cube. It does not roll, but by turning it on an edge, it flops back down onto the table. You can do this in two different directions, leading in fact to a group of symmetries of order 24 (i.e., containing 24 elements). The fact that there are so many is based on the fact that these motions are not *commutative*, i.e., the order in which they are carried out is important. This group is known to mathematicians as the *symmetric group on four letters*, and can be described as a group of permutations of four objects (usually referred to as letters).

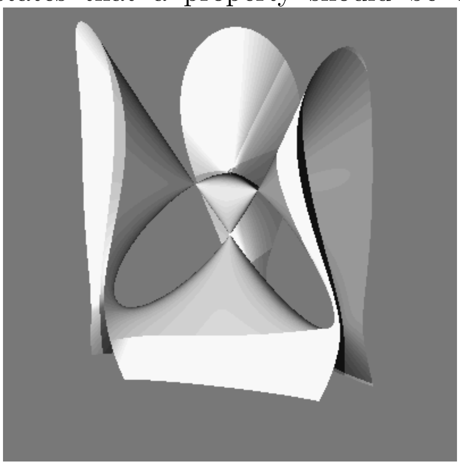


Figure 8: The Cayley cubic surface

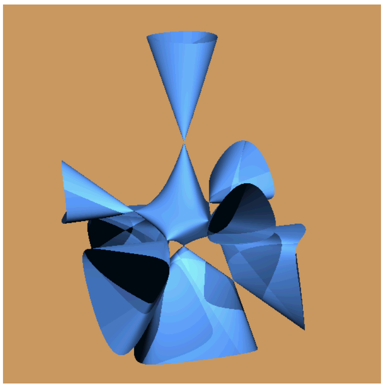
We now present some surfaces with nice symmetry. First of all, a special surface with the group Σ_4 as symmetry group: the *Cayley cubic*. This surface was first studied by Arthur Cayley around 1850, and Felix Klein had a plaster model of it prepared for the Chicago World Fair in 1882?, thus initiating the creation of plaster models to visualize mathematical objects. In Figure 9, this surface is depicted.

The fact that this surface has the permutation group of four letters as its symmetry group is easily seen from its projective equation, which is:

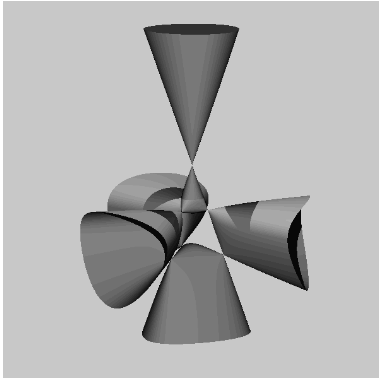
$$\frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = 0,$$

which is the equation of an algebraic surface perhaps contrary to appearances: just multiply through by $x_0x_1x_2x_3$.

This surface is, by the way, also of particular interest as it is the unique cubic surface which has four ordinary double points. It is not difficult to visualize the symmetries of this surface which consist of the *rotations* of the tetrahedron whose vertices are the singular point. It is more difficult to envision the symmetries which arise from a *simple permutation* of two of the (homogenous) coordinates, which is a *reflection* on a certain plane. These planes are those which contain two of the vertices and pass midway between the other two. These symmetries can be described as follows: suppose that we place a mirror exactly at the location of this plane, and you look at the mirror from one side. Then what you see is *exactly* the same thing you would see if we had put, instead of the mirror, a transparent pane of glass.



(a) The Hessian of the Cayley cubic



(b) The Hessian of the Clebsch cubic

Figure 9: Hessian varieties of cubics surfaces; these are quartic surfaces

Another very beautiful cubic surface, the unique such which has the symmetric group on *five* letters as its symmetry group, is the *Clebsch cubic*, also known as the *diagonal cubic surface*. In this case the surface is smooth, and one does not "see" the symmetry just looking at the surface, which is depicted in Figure 10. Again, the symmetry can be seen by looking at the projective equation, this time with a twist. One describes the surface in $\mathbb{P}^4$ instead of $\mathbb{P}^3$, in which it lies in a particular *hyperplane*, which cuts out of $\mathbb{P}^4$ a $\mathbb{P}^3$.

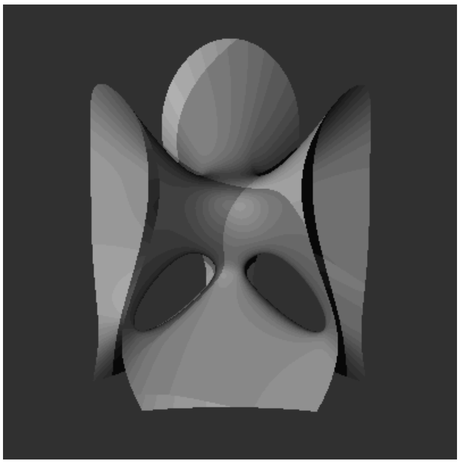


Figure 10: The Clebsch "diagonal" cubic surface

The equation is then

$$\mathcal{C} = \{y_1^3 + \cdots + y_5^3 = 0 = \sum y_i\}; \tag{2}$$

the equation clearly stays the same (is *invariant* under an arbitrary permutation of the y_i. Some other surfaces of interest for their symmetry groups follow. There is an interesting notion of *covariant* of hypersurfaces, in particular also of surfaces. These are surfaces whose equation is obtained in some manner from a given equation in such a way as to preserve all symmetries. A prime example is that of the *Hessian variety*, whose equation is easily written down: let $f(x_0, x_1, x_2, x_3)$ be the defining equation of surface, let $\partial_i f$ denote its partial derivative with respect to x_i, and form the *Hessian matrix*:

$$\{\mathrm{Hess}(f)\}_{i,j} := \partial_i \partial_j f.$$

This is a four by four matrix, in the case of our cubic surfaces above for example with entries which are linear forms in the variables x_i. Then one forms from this matrix the *determinant*, which is then a single polynomial of degree $4(d-2)$ for a surface of degree d, so for the cubic surfaces above this is a *quartic*. We have some nice pictures of the Hessian of the Cayley and Clebsch cubic, see Figure 9. As mentioned above, the Hessian is a covariant of a polynomial, so it has the same symmetry as the original surface.

A further example of covariants is the so-called *dual variety* of a hypersurface. See my book "The Geometry of some Speical Arithmetic Quotients", Springer Lecture Notes in Mathematics **1637**, Springer-Verlag 1996, section B.1.1.6 for more details on this notion. The dual variety is defined to be the union of all hyperplanes of the ambient projective space (in this case, all planes in three-space) which are *tangent* to the hypersurface. Here one uses the notion that the set of *all* hyperplanes of projective space is itself a projective space (duality principle alluded to above). Thus, starting with a given surface, we get another surface, the dual. In general, even for surfaces of low degree the degree of the dual is quite high, namely

$d(d-1)^2$, where d denotes the degree of the surface. For example, the dual of a smooth cubic has the degree $3(2)^2 = 12$. However, if the surface has singularities, this reduces the degree of the dual. In the case of nodes, each node reduces the degree by two. Hence, the dual of the Cayley cubic is $3(2)^2 - 2 \cdot 4 = 4$, which is a quartic surface (surface of degree four).

Being the dual of a very unique cubic surface, this quartic is very unique. It turns out to be the unique quartic surface which has three singular lines which meet at a point (here the set of singular points is not a finite set of points, but consists of a union of lines, which is referred to by saying the "singular locus is of dimension one"). It can be shown that this quartic surface has the following property: it is the projection into three-space of the *Veronese surface* in $\mathbb{P}^5$. This is the unique surface in $\mathbb{P}^5$ whose variety of chords (a chord is a line intersecting the surface in two points; in general a line in five-space will not meet a given surface at all) is a proper subvariety of $\mathbb{P}^5$. It is in fact a cubic hypersurface. The Veronese is also (another favorite of mathematicians: give as many different descriptions of given objects as you can) the image of $\mathbb{P}^2$ under the so-called Veronese map $(s,t) \mapsto [1 : s : t : s^2 : st : t^2]$ (the former coordinates are affine coordinates on $\mathbb{P}^2$, the latter are homogenous coordinates in $\mathbb{P}^5$). The image is a surface of degree four, and this degree is preserved under projection. As the Cayley cubic has Σ_4 as symmetry group and as the dual variety is a covariant, also the dual surface has this symmetry.

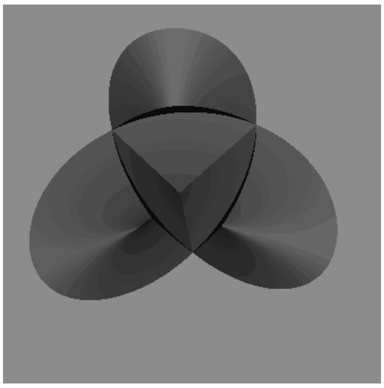


Figure 11: The dual of the Cayley cubic, a quartic surface which is the projection of the Veronese surface in $\mathbb{P}^5$

So far, we have essentially discussed only two groups as symmetry groups: the symmetric groups on four and five letters, Σ_4 and Σ_5, respectively. Actually, the interesting groups which can occur are not too numerous. The reason is that if we have a surface with the given symmetry group, then this group also acts on the ambient projective space, and *these* groups are highly restricted, there just are not too many of them. In the theory of groups, one has certain kinds of building blocks, the so-called *simple* groups. There is a short list of simple groups which act on the projective three-space, and thus there is a short list of simple groups which can occur as the symmetry groups of surfaces embedded in projective three space.

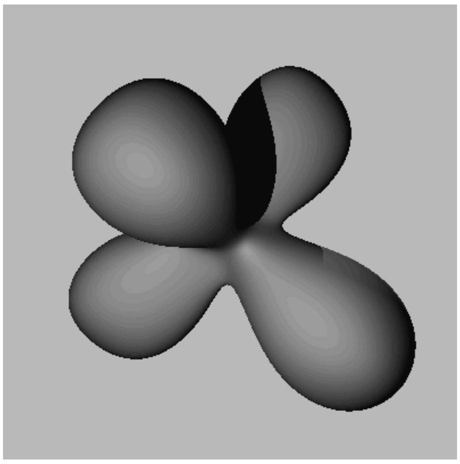
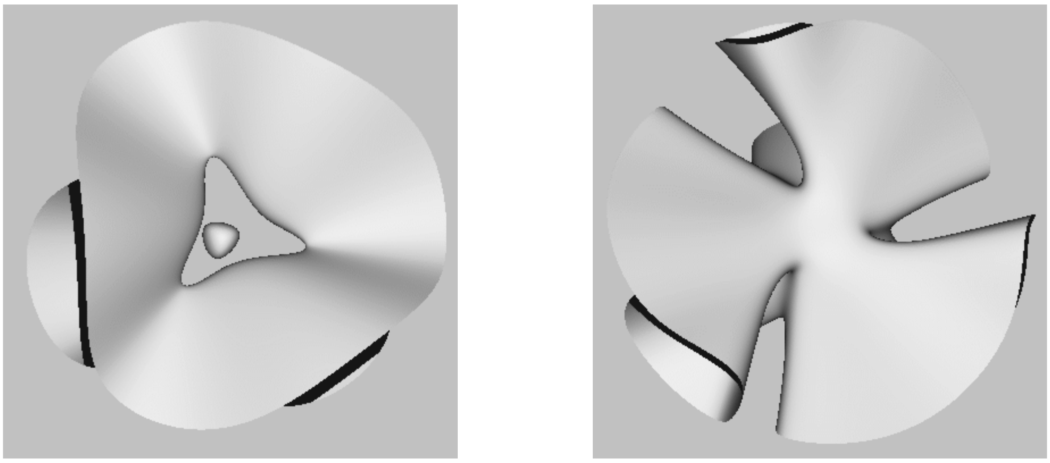


Figure 12: The invariant of degree 8 of the simple group of order 11,520

In addition to those already mentioned, we just mention two further ones. As to the first, whose symmetry group has order (number of elements) 11,520, Figure 12 suggests it just also has a symmetry group which permutes four letters, in this case the four "lobes" in the picture. In fact, we *cannot* see all the symmetries at once. This is an example where the real pictures we are drawing are misleading; in this case the symmetries themselves are complex! The symmetry group is a so-called *unitary reflection group*, which is generated by *complex reflections*, rather that real ones. Even experienced mathematicians have difficulty envisioning this.

As to the second, this is a wonderful group of order 168, which acts on the projective plane and on projective three-space as well. In mathematical terms, it is the simple group $G = PSL(2, \mathbb{F}_7)$. We display pictures of the unique invariants of degrees 4 and 6 in Figure 13. The equation of the degree 4 invariant is simple enough to write down here: it is $t^4 + 6\sqrt{2}xyzt + 2(y^3z + z^3x + x^3y) = 0$, the parenthesized expression being the equation of the famous *Klein curve* in the projective plane (this is the unique invariant *curve* of degree four under the action of G_{168} on the projective plane we have already mentioned). For the mathematicians let us add a few details on this remarkable curve. It is, on the one hand, the quotient of the upper half-plane by a principal congruence subgroup in the arithmetic triangle group $(2,3,7)$. The latter group is generated by three elements of orders 2, 3 and 7, and the principal congruence subgroup arises from a certain subalgebra in a division quaternion algebra (a maximal order in this algebra is the arithmetic group $(2,3,7)$ in a different guise). On the other hand,



(a) The invariant of degree 4

(b) The invariant of degree 6

Figure 13: Two invariants of the simple group of order 168

the same curve is the compactification of the quotient of the upper half-plane by the principal congruence subgroup of level 7 in $SL(2,\mathbb{Z})$. This is an example of a *Janus-like algebraic variety* (see B. Hunt & S. Weintraub, *Janus-like algebraic varieties*, J. Diff. Geo. **39** (1994), 509).

3 Singularities

The dialog continues.

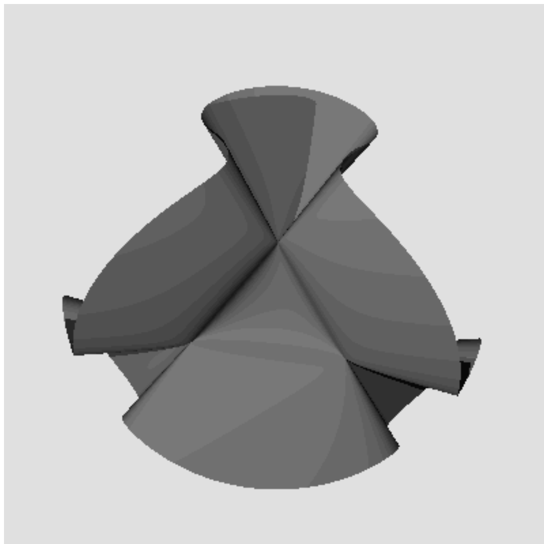
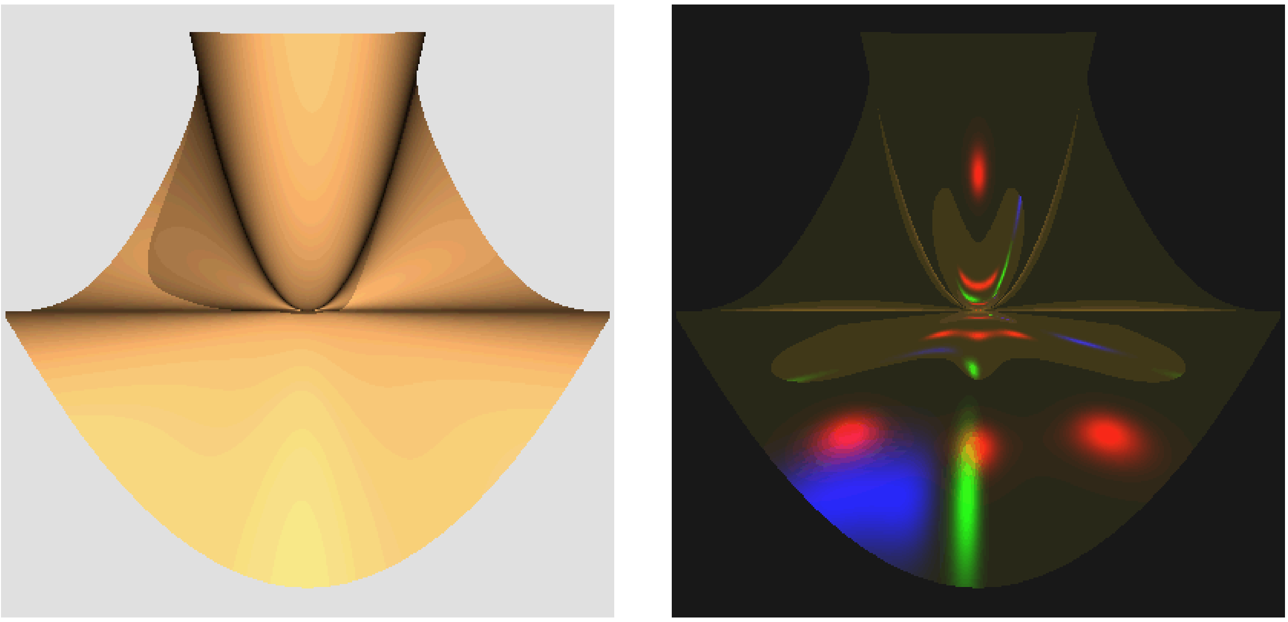


Figure 14: The cubic surface with three singularities of type A_2

> Mathematician: "The next thing I would like to explain to you is the notion of a singularity. Suppose you go to take a seat at the market place in some European city, and you observe people, thinking about a nice motive for your next painting. Do all people you see leave the same impression on you?"
> Artist: "Of course not. Some people are more interesting than others."
> Mathematician: "And what is it that makes some people more interesting?"
> Artist: "Some people wear more interesting clothes, some people have a very particular and unusual way of moving about. Some people are pretty, while other are rather boring. And every once in a while somebody comes along that really catches your eye."
> Mathematician: "Right, that is somebody which is truly unique and one-of-a-kind. Now, thinking instead of people just of objects of some sort, the same is also true. This one-of-a-kindness is what mathematicians refer to as *singular*, and an object which has this property is called a *singularity*."

The objects which catch most of the mathematicians interest, just like I suppose most people's, are those objects which are different and more special than all others. These are the objects which are referred to as



(a) The E_6 cubic surface

(b) The same surface, made of mirror glass, with three sources of light, one red, one green and one blue

Figure 15: The "most" singular cubic surface

singular ones, or, in come cases, just as singularities. For example, among all surfaces of degree three, we have seen two of the most singular, in the sense of being the most special. One of them (the Cayley cubic) is also the most singular in the sense of having the most singularities, where here singularity refers to *singular points* on that surface. There are two basic kinds of questions which mathematicians are obsessed with. These are "Does it exist?" and "How many are there?". In the present context, we ask whether and how many singularities exist, and also how many *kinds* of singularities are there. This is a precise mathematical notion, which is based on the notion of *equivalence*, here equivalence of singularities. It turns out that there are in fact not so many singularities, at least not that can occur on surfaces in three-space (these are termed *two-dimensional hypersurface singularities*). In particular, there is a short list of the possible singularities which can occur on cubic surfaces (the higher the degree of a surface, the more nasty kinds of singularities which can occur). For example, there is a unique cubic surface which has four of the simplest singularities, termed A_1-singularities: this is the Cayley cubic introduced earlier. There is also a unique cubic surface which contains *three* of the next-worst singularity, called (as you might suspect) the A_2. This surface, whose projective equation is quite simply $x_0^3 + x_1x_2x_3 = 0$, is pictured in Figure 14. A remark for those familiar with singularities: this is the most singular *semistable* cubic surface.

In Cayley's famous paper "A memoir on cubic surfaces", a classification of the different types (possible singularities) of cubic surfaces was given. He came up with a list of 23 cases. The most singular of which has a beautiful singularity with the name E_6. Especially in the case of cubic surfaces this is highly interesting, as the singularity of this type is related to cubic surfaces in curious ways. For the mathematicians in the audience, a brief explanation is as follows. The set of 27 lines has a very particular *combinatorial* structure, in that the set of "permutations" of the 27 lines that also preserve the incidence structures is a very special subgroup G of the permutation group Σ_{27}. It is a group of order 51,840 (which has a normal subgroup of index two which is *simple*); this is the *Weyl group* of the Lie algebra of type E_6. The combinatorial structure of the set of Weyl chambers of the Lie algebra is identical to a similar kind of symmetry related to the singularity (versal deformation space), and a deep conjecture of Grothendieck, proved later by Brieskorn, relates the versal deformation space with the Lie algebra *directly.* Still, the relation is kind of mysterious, and is a typical example of the kind of wonderful mysteries nature has in store for future generations of mathematicians who want to explore.

We have depicted the surface in Figure 15, in the second picture as a mirror. This emphasizes the special shape of the object, and is just plain beautiful in its own right. The set of most of the interesting singularities which occur are contained in the images of Figures 16 and 17. The last one in his list is in Figure 18; this is the so-called *Whitney umbrella*, a very well-known singularity which is often used to explain and test concepts. Note that this last example is the only case in which the singular locus is one-dimensional.

Figure 18: The last cubic surface in Cayley's list, also known as "Whitney's umbrella"

In the theory of singularities there are two basic notions of how to "improve" a singularity, in the sense of making a singular point "less" singular: *deformation* and *resolution*. At least the first can be made very easy to visual using movies, showing how a singular point arises as a case of a very special set of parameters from a situation in which there are either no singular points at all, or there are singular points which are much "less singular". Another notion in algebraic geometry is that of *degeneration*, which is what happens when we start deforming something and then run into a set of parameters which are so special that the surface completely changes its structure; this again can be beautifully visualized with movies. We now give a few examples.

3.1 Maximal numbers of singularities

We have already mentioned cubic surfaces. As far as surfaces of higher degree are concerned, some famous beautiful pictures are of surfaces having a maximal number of *ordinary double points*, that is, singularities of type A_1. The following pictures from the Gallery show such for degrees $d = 4, 5, 6, 8$ and 10. We should mention here that for higher degrees there are indeed *bounds* on the number of ordinary double points they can have (and of course also for worse singularities), but it is not in general known whether this known bound is actually obtained.

1. **Quartics:**

A quartic surface (i.e., of degree 4) can have at most 16 A_1 singularities. There is in fact a three-dimensional family of such surfaces, known as *Kummer* surfaces from the research of Edward Kummer in the last century. This family of surfaces has ties to several other interesting areas of mathematics, in particular it is closely related to the family of *Jacobians* of curves of genus two, or equivalently, of *Abelian surfaces*. The latter in turn are related to many areas of topology, geometry and number theory.

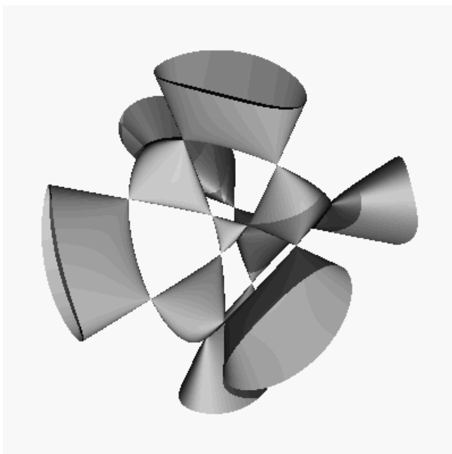
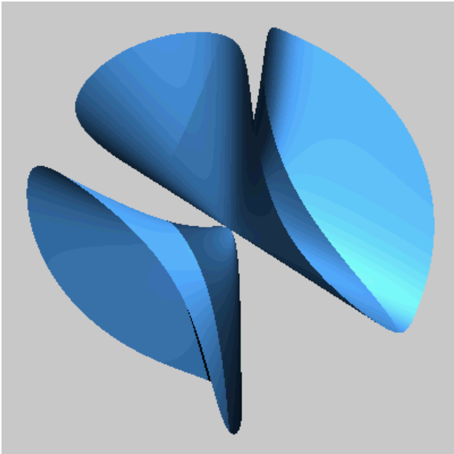
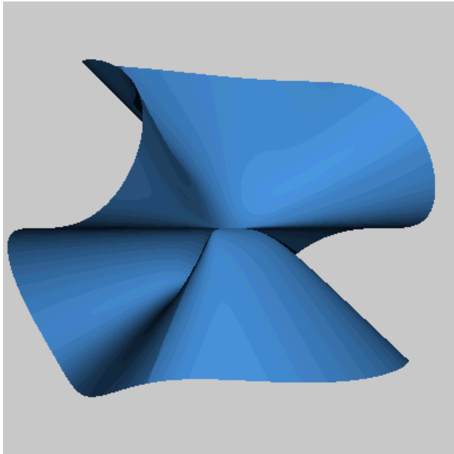


Figure 19: A Kummer surface

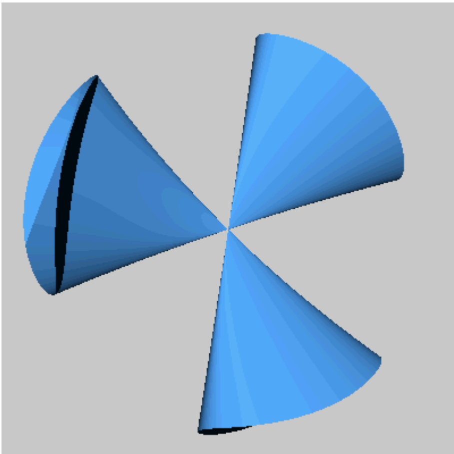
Each of the surfaces is beautiful to look at. There is a delightful combinatorial structure related to them, called the *Kummer* 16_6-*configuration*. This configuration consists of, in addition to the 16 ordinary double point, 16 planes, and they share the wonderful property that *in each of the 16 planes, 6 of the double points lie, and conversely, through each of the 16 double points, 6 of the 16 planes pass.* This configuration is in turn related to the even theta functions of genus two, one of the favorite topics of inquiry late in the nineteenth century.



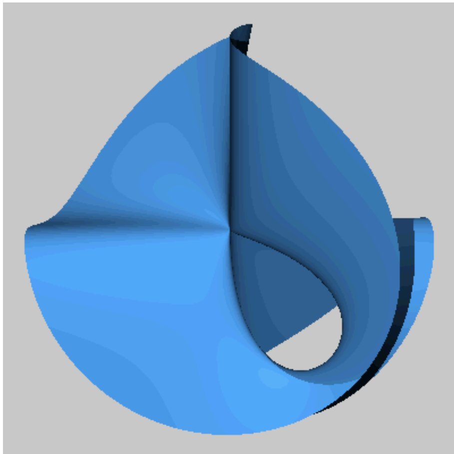
(a) $xz + (x+z)(y^2 - x^2) = 0$



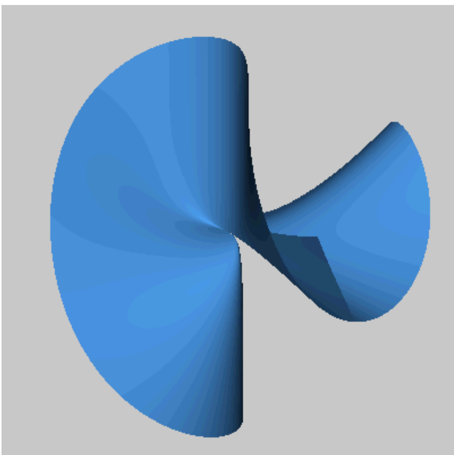
(b) $xz + y^2z + x^3 - z^3 = 0$



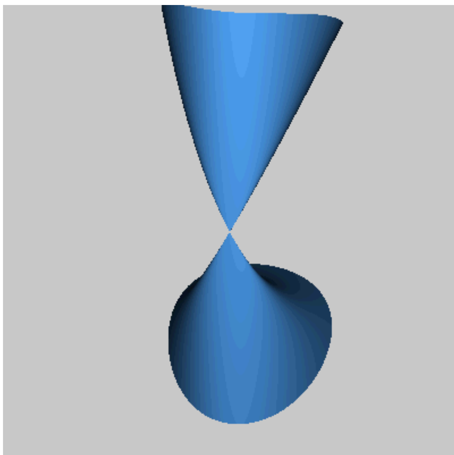
(c) $(x+y+z)^2 + xyz = 0$



(d) $xz + y^2(x+y+z) = 0$

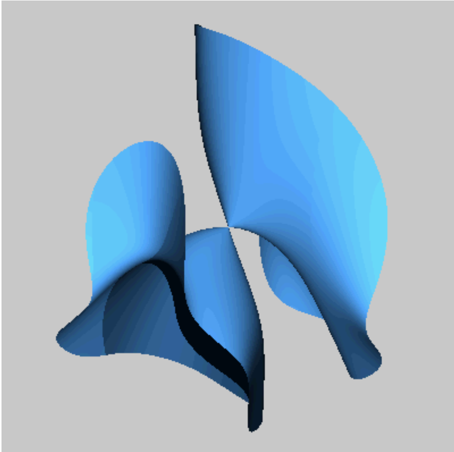


(e) $xz + y^2z + yx^2 = 0$

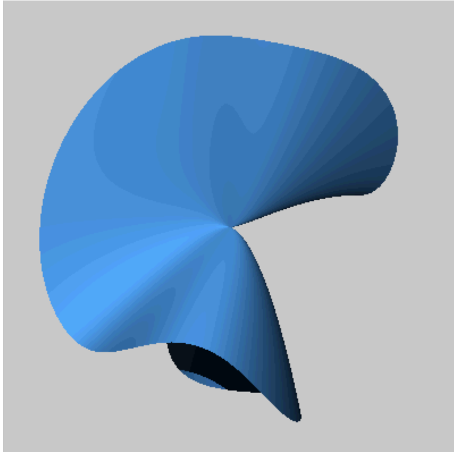


(f) $x^2 + xz^2 + y^2z = 0$

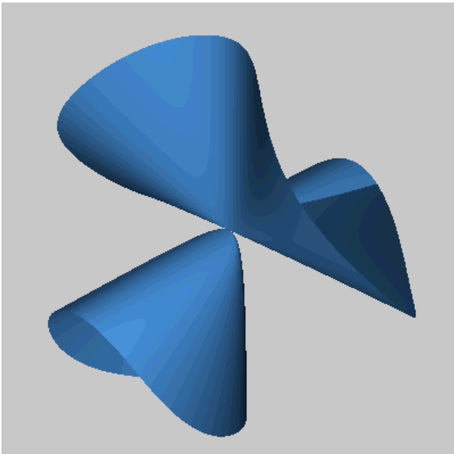
Figure 16: Some of the singular cubic surfaces from Cayley's list



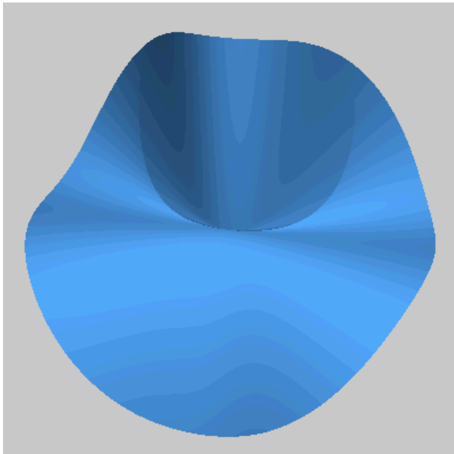
(a) $(xy + xz + yz) + xyz = 0$



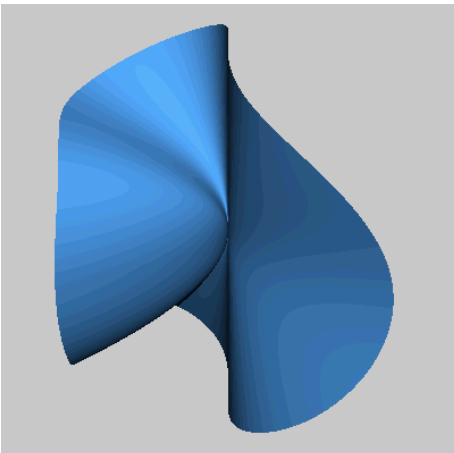
(b) $xz + xy^2 + y^3 = 0$



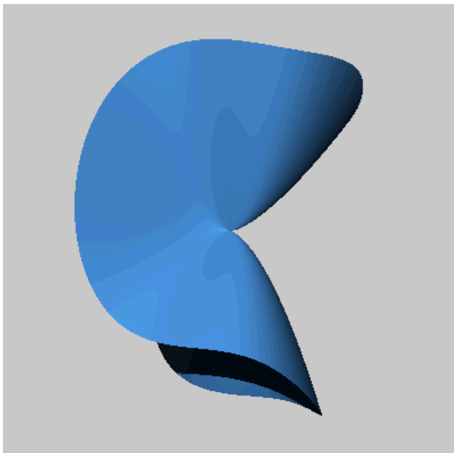
(c) $xz + (x + z)y^2 = 0$



(d) $xz + y^2z + x^3 = 0$



(e) $x^2 + xz^2 + y^3 = 0$



(f) $xz + y^3 = 0$

Figure 17: Some of the singular cubic surfaces from Cayley's list

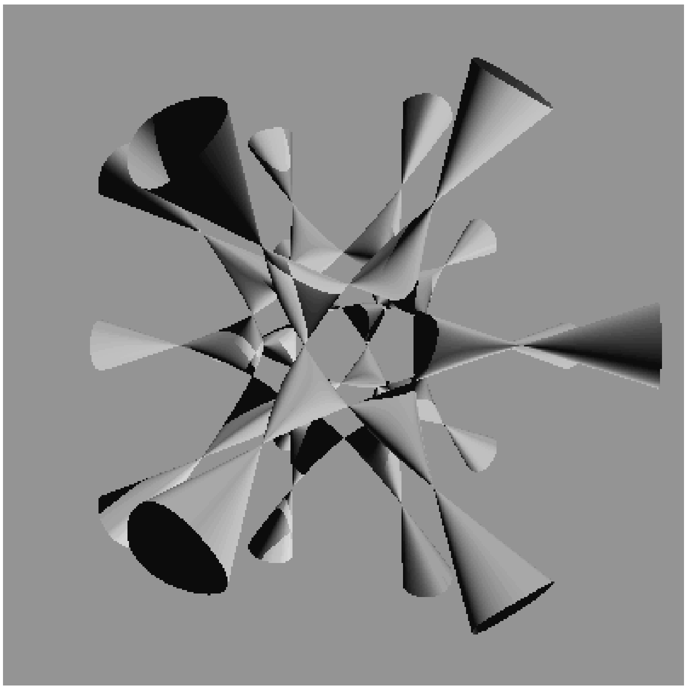


Figure 20: Barth's sextic with 65 nodes

We show in Figure 19 one of the Kummer surfaces.

2. **Quintics:**

In the case of quintics it is known that the maximal number of ordinary double points they admit is 31, although it was not clear for a long time whether such quintics actually exist. In fact, they do, and here is a picture of a surface which was derived by Duco v. Straten, Stephan Endrass and Wolf Barth (we will be hearing about Barth a couple of times in the sequel, as he has found many of the "records", i.e., surfaces with maximal known numbers of double points). In order to make the visualization easier, we have added mirrors in the back and below, so the viewer may see the surface from more than one side.

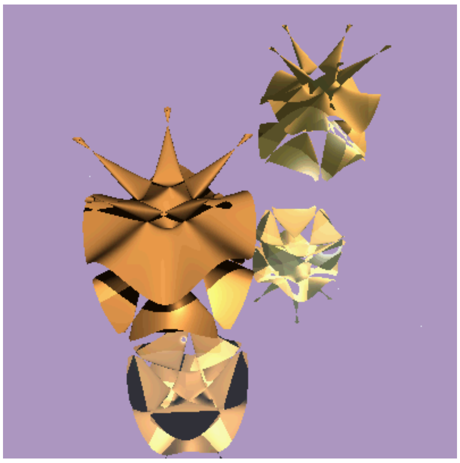


Figure 21: A Togliatti surface, a quintic with 31 double points

The equations of many surfaces with a large number of ordinary double points can be found, together with pictures, which were made with the drawing tool SURF developed by Endrass, in his article *Flächen mit vielen Doppelpunkten*, DMV Mitteilungen **4** (1995), 17-21. In many cases one finds that these surfaces not only have a large number of double points, but they also have a relatively large symmetry group, making them interesting from this point of view also. However, finding the equations of these surfaces is in general a very difficult problem, and requires not only sound knowledge in the theory of surfaces but also a bit of ingenuity.

3. **Sextics:**

   Here we just present the picture of the famous sextic found by Barth, which has 65 ordinary double points, the maximal number which can occur, in Figure 20.

4. **Octics:**

   There is a series of beautiful surface with many double points constructed by Čmutov. Of these, we have a picture of an octic which has 112 nodes in the upper picture of Figure 22. The maximal number is 168, but we find this surface to be more interesting to look at.

5. **Surfaces of degree ten:**

   As a final example we display the "most incredible of all" of the set of surfaces we are discussing: a surface of degree ten with 345 nodes! This surface was again discovered by Barth, and shows how much symmetry a surface which is this special can have. Looking at the picture it is not difficult to imagine that it is invariant under the symmetries of an icosahedron in three-space; this group is a subgroup of index two (meaning that its order is just half of) of the symmetric group Σ_5 which was already discussed in relation with the Clebsch cubic surface.

3.2 Deformations

During the talk at the colloquium, we showed movies which impressively demonstrated the process of deforming singularities to smooth them, or at least to make them "better". Here we show a sequence of pictures which demonstrate this to some extent.

As a first example, we consider the so-called A_2-singularity. Such a singularity is given mathematically by some equation, in this case it is $x^2 - y^2 + z^3 = 0$. The fact that the point $(0,0,0)$ is a singular point follows from the fact that all partial derivatives of the function, which are $2x$, $-2y$ and $3z^2$, vanish at that point. In order to "smooth" the singularity, one simply perturbs this equation such that no longer all partial derivatives vanish. This is achieved for example by adding a constant term, whose precise value is indeterminant:

$$x^2 - y^2 + z^3 + t = 0,$$

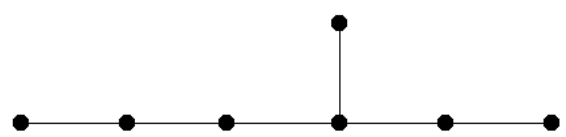
which describes a family $\{S_t\}$ of smooth surfaces for which only the surface S_0 for the parameter value $t = 0$ is singular, all others are smooth. In the upper series of three pictures in Figure 23 we show three surfaces in the family, giving an impression of what the smoothing of the singularity looks like.

On the other hand, one can deform the singularity by keeping the point singular, by lessening the degree of singularity. In this case this means we deform the A_2-singularity to a A_1. The equation for an A_1-singularity is just $x^2 - y^2 + z^2 = 0$, so we deform by the family

$$x^2 - y^2 + t \cdot z^3 + (t-1)z^2,$$

which is an A_2 singularity for $t = 1$, but an A_1 singularity for $t < 1$. This is depicted in the bottom series of Figure 23.

Once the principal is clear, one can just observe the kind of degenerations which can be constructed. As a particularly interesting example, we show also the deformation of a E_7 singularity to four ordinary double points ($4A_1$). Both the singularity of type A_n and those of types E_N, $N = 6, 7, 8$ are a type of singularity which are called *rational double points* (of which the simplest, A_1, is called the *ordinary* double point). For these singularities, the possible deformations of one type to another is clearly revealed upon inspection of the so-called *Dynkin diagram* of the singularity. In the case of A_1, this is also the simplest possible graph, consisting of just one vertex. For E_7, the diagram is:



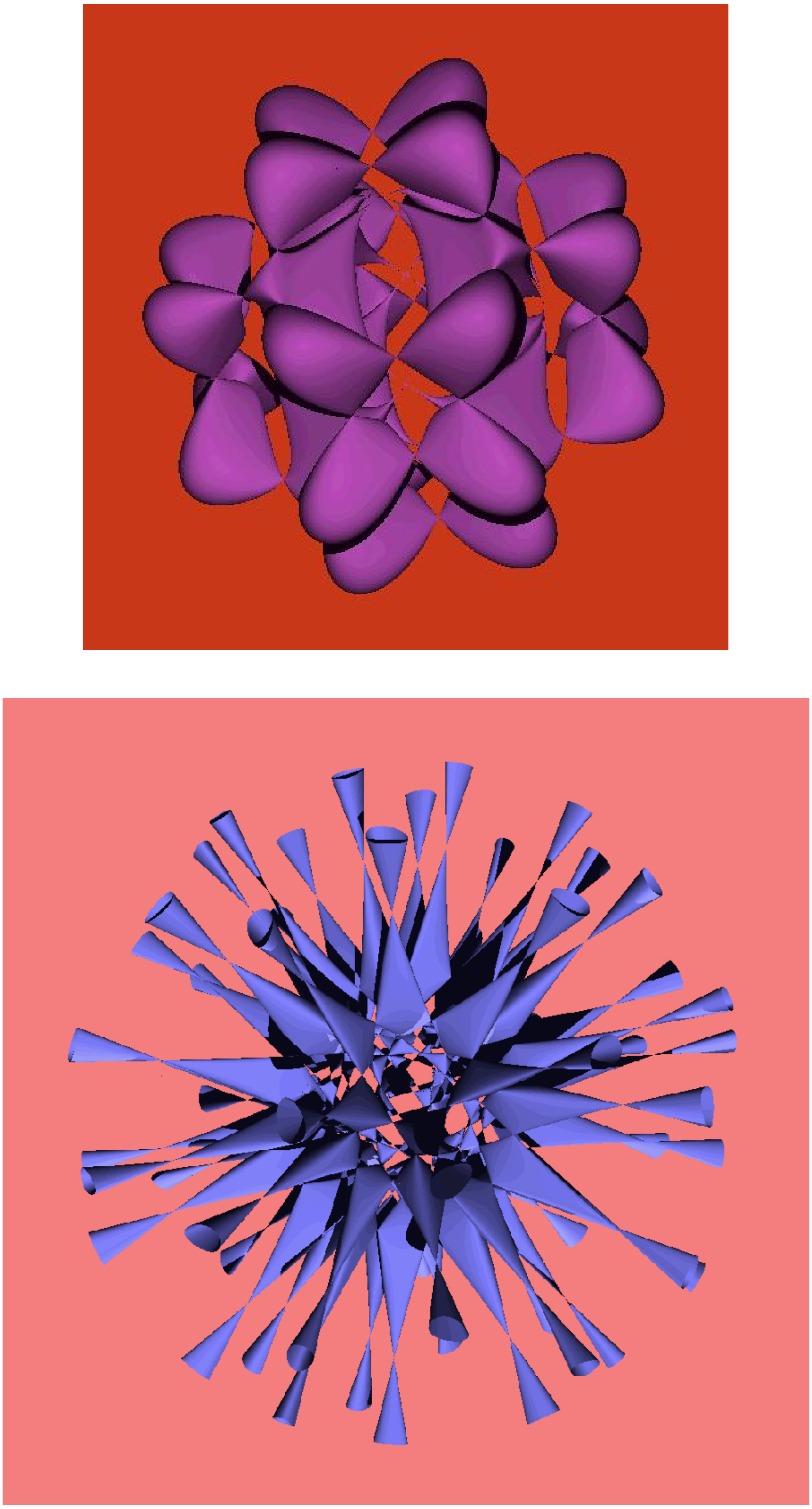


Figure 22: A Čmutov octic with 112 nodes and the surface of degree ten with 345 nodes

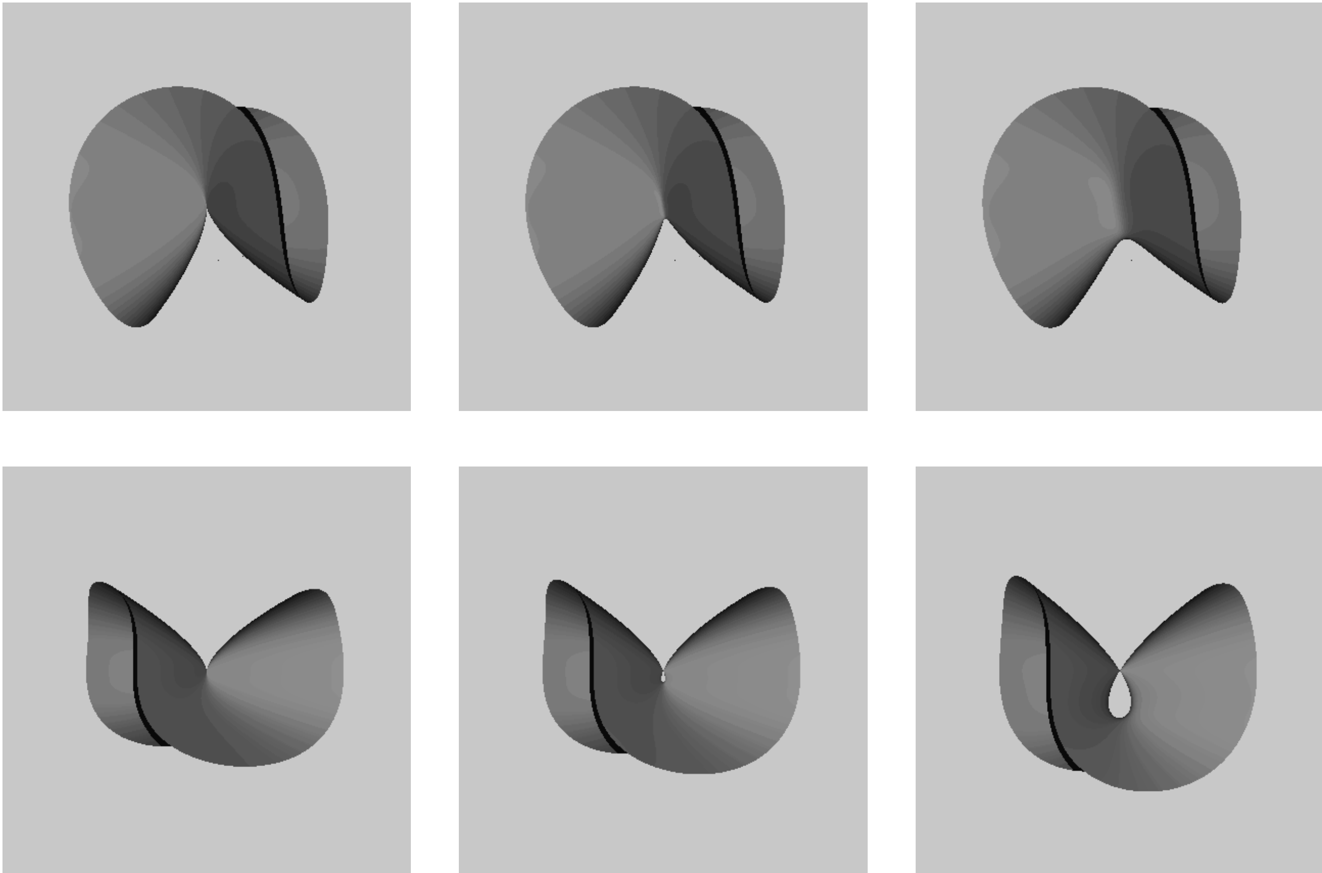
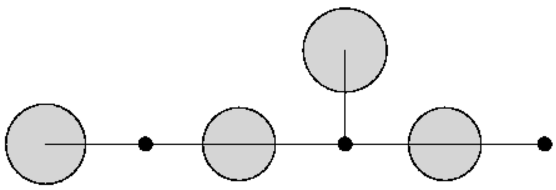


Figure 23: The smoothing of a singularity of type A_2, and the deformation from the A_2 singularity to the A_1

The possible ways in which E_7 can be deformed is equivalent to the possible *disjoint subgraphs* of the E_7 graph. In the case at hand, we can find a total of four disjoint vertices, indicating the pictured deformation:



3.3 Degenerations

What we have been considering up to now is a kind of mild change. However, in mathematics also catastrophic changes are of great interest. For example, a smooth surface or a surface with *isolated* singularities of the kind we have been discussing up to now may, for particular values of the parameters, break up into several pieces. This is what is known in algebraic geometry as *degenerations*. We give some examples of these, where the families are of particular interest. During the talk we showed movies of a family of desmic surface (surfaces of degree four, i.e., quartics), which have 12 ordinary double points, which for special parameter values degenerate into *four planes* (this is also a quartic!). In Figure 26 we show a couple of the frames of this movie.

Another movie we showed is of another family of quartics, which again degenerates for special parameter values to four planes, but for which there are two different types of pictures (over the reals, but in fact

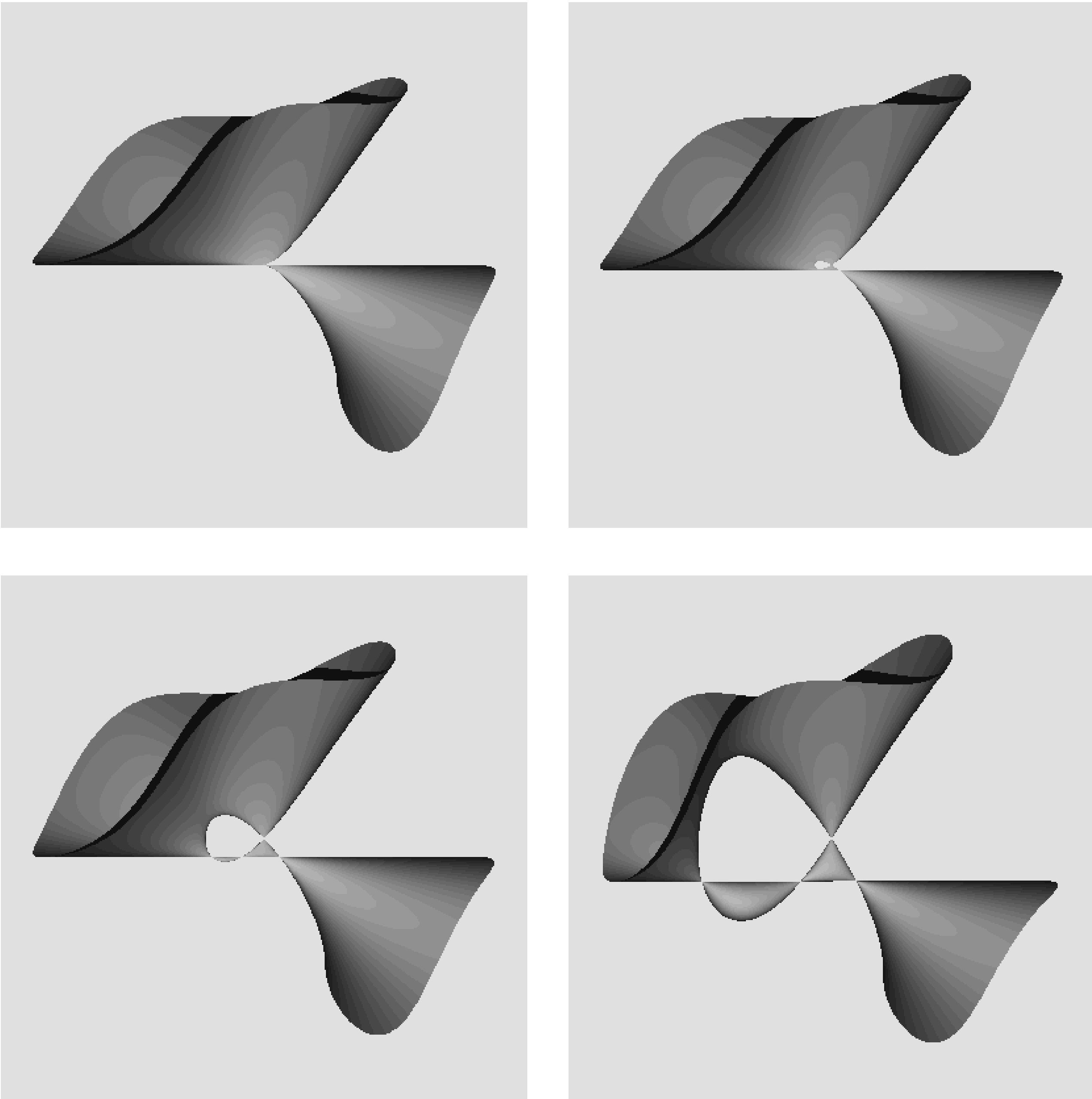


Figure 24: A deformation of the singularity of type E_7 to four ordinary double points

these are also different over the complex numbers too). On the one hand we have a famous surface called a *Steiner surface*, on the other a famous set of quartics known as *Kummer* surfaces, which we have mentioned already above. Figure 25 shows a few frames of this movie.

As to the desmic surfaces, they are also quite interesting objects. Now more for the mathematicians in the audience, let me briefly comment on this. The Kummer surfaces which were mentioned above are the *Kummer varieties*, i.e., the quotient by an involution of Abelian surfaces which are *generic* in the sense that they are not the product of two Abelian curves (known in general as *elliptic curves*). The Kummer surfaces have 16 nodes, which are the images of the 16 so-called *2-torsion points* on the Abelian surface, that is points which, in the natural group structure of the Abelian variety[2], are 2-torsion, meaning that adding them to themselves results in zero. The desmic surfaces are also Kummer varieties, but now they are the quotients

[2]An Abelian variety is by definition a smooth, projective variety with a group structure.

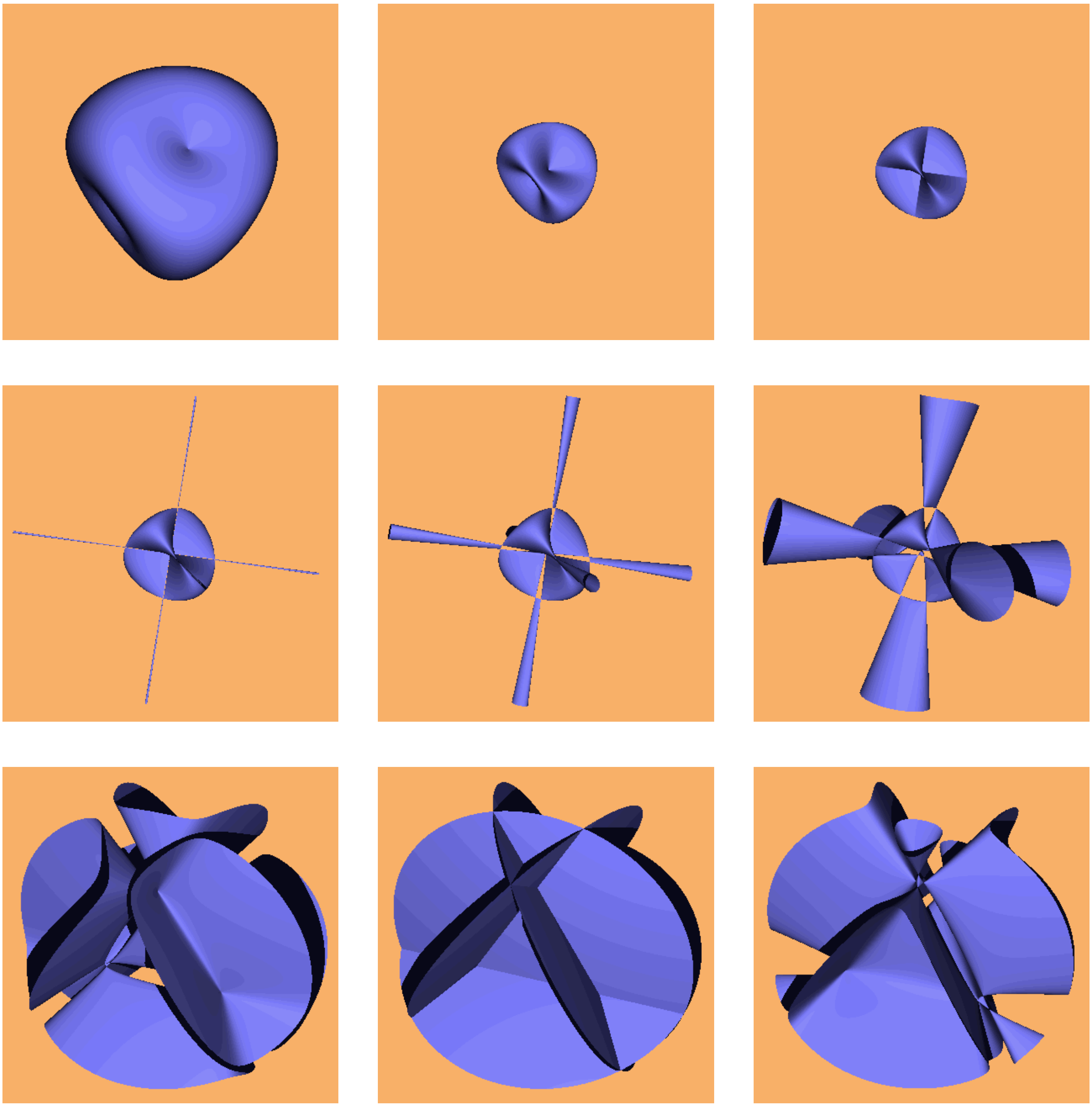


Figure 25: A degeneration: first the Steiner surface degenerates into a Kummer surface, the latter then degenerates into four planes

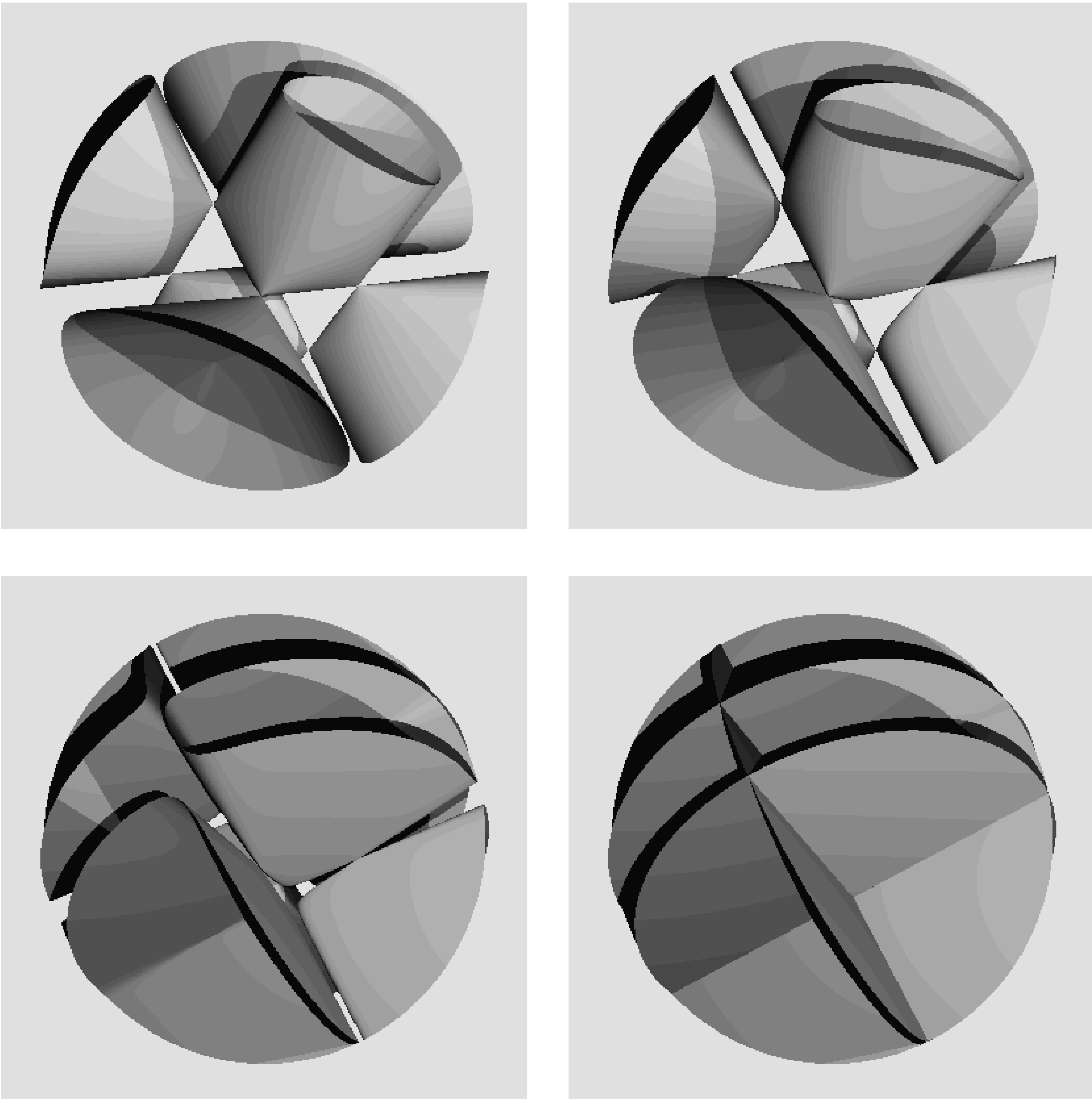
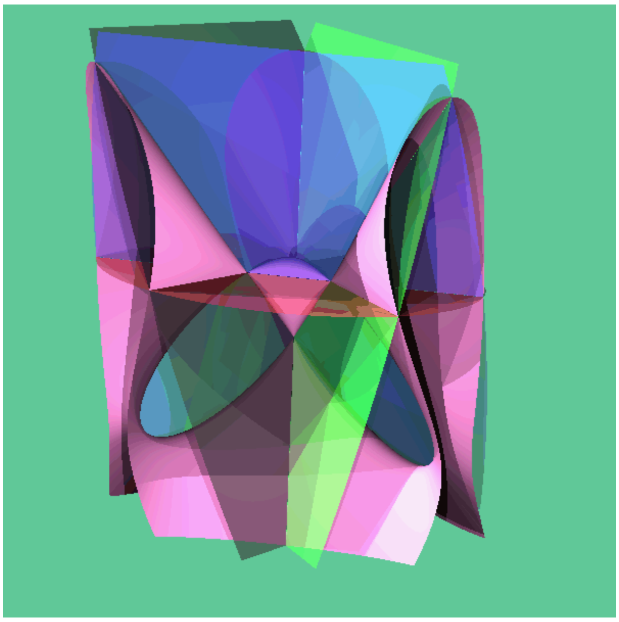
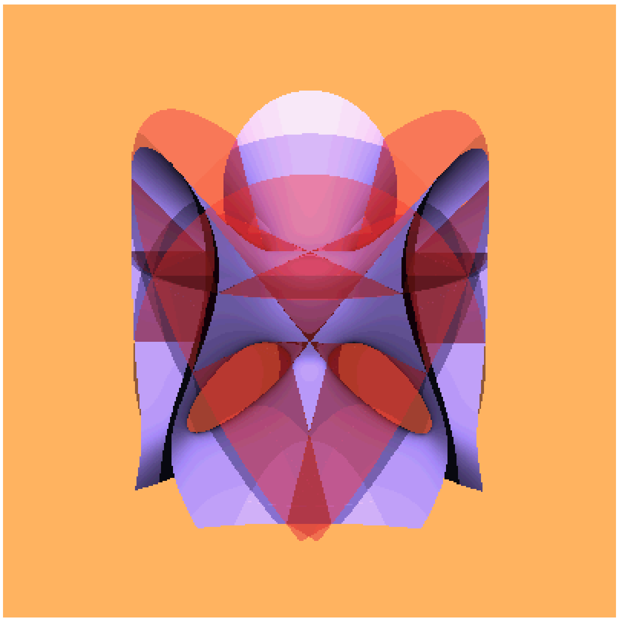


Figure 26: A degeneration of desmic surfaces to the union of four planes



(a) The Cayley cubic with some tritangents and lines



(b) The Clebsch cubic with some tritangents and lines

Figure 27: Tritangents and lines on cubic surfaces

of Abelian surfaces which *are* products of elliptic curves, more precisely, the product of two copies of the same elliptic curve, $E_\tau \times E_\tau$. Since the moduli space of elliptic curves is just one-dimensional, this explains that there is only a one-dimensional family of desmic surface. The desmic surfaces on the other hand have 12 nodes instead of 16, but there are 16 lines which lie on them. The name desmic comes from the fact that their equations can be written

$$a\Delta_1 + b\Delta_2 + c\Delta_3,$$

where the Δ_i form a system of *desmic tetrahedra*, which means that each of the three is in perspective with respect to the remaining two (this is not easy to imagine, it means more precisely that there are four centers of perspective, and these four points are the vertices of the remaining tetrahedron). There are 16 lines through which a face of *each* of the Δ_i pass, and these 16 lines lie on each of the desmic surfaces. The twelve vertices of the Δ_i are the nodes of the desmic surfaces. The twelve lines play the role of the 16 nodes on the Kummer surfaces (points of order two), while the nodes are images of special curves on the corresponding products $E_\tau \times E_\tau$. For more details, see my book "The Geometry of some Special Arithmetic Quotients", Springer Lecture Notes **1637**, Springer-Verlag 1996, section B.5.2.3.

4 Enumerative geometry

The dialog continues.

> Mathematician: "The final thing I wanted to try to explain, and by far the most difficult, is what we call enumerative geometry. It is an incredibly beautiful topic, but requires a certain amount of mathematical background to really understand. But here goes. Consider a building which has, instead of a usual roof, some curves surface, as for example the Olympic stadium in Munich. This is an example of a curved surface. Can you imagine it?"
> Artist: Oh, of course. Almost everything I paint consists of pieces of curved surfaces like that."
> Mathematician: "Wonderful! That is exactly what I mean. Now such a surface is not "strait",

but it might still happen that certain lines could lie on such a surface."
ARTIST: "For very special surfaces and lines, I suppose it could."
MATHEMATICIAN: "Exactly. The word very special is important. Now what a mathematician likes to do is to find such surfaces where this can happen, and to *count* the number of such lines there are."
ARTIST: "That reminds of a geometry course I once took."
MATHEMATICIAN: "Think of something slightly different. You are making a perspective drawing, and you have a globe in the foreground, and want to draw lines of perspective from it. Which lines would you choose?"
ARTIST: "One which meets the globe at the top and the bottom, in just one point."
MATHEMATICIAN: "You have real mathematical talent. Those two lines are the ones we say are *tangent* to the globe, and again, we can consider objects other than the sphere and consider the same question. This is also a problem in enumerative geometry."

The subject of *enumerative geometry* is concerned with counting problems, counting objects of which there are only finitely many in some given configuration. We have already met some examples above: there are 16 planes which meet a Kummer surface six at a time in its 16 ordinary double points. There are 16 lines which lie on a desmic surface. This kind of result is definitely one which will be valid only over the complex numbers. In real algebraic geometry, there will only be certain configurations and examples for which all, say 16 planes of a Kummer surface, are also real and can be visualized.

We will show some nice pictures for the easiest examples of such phenomena, which is the case of cubic surfaces. A lot has been written about this subject, but, as my experience tells me, it is still the most accessible for non-mathematicians. The magic numbers here are 27 and 45; 27 is a nice number, being the first odd number which is a cube. In the situation here, however, 27 more naturally arises as $12 + 15$. Anyhow, the statement is that:

There are exactly 27 lines on a smooth cubic surface.

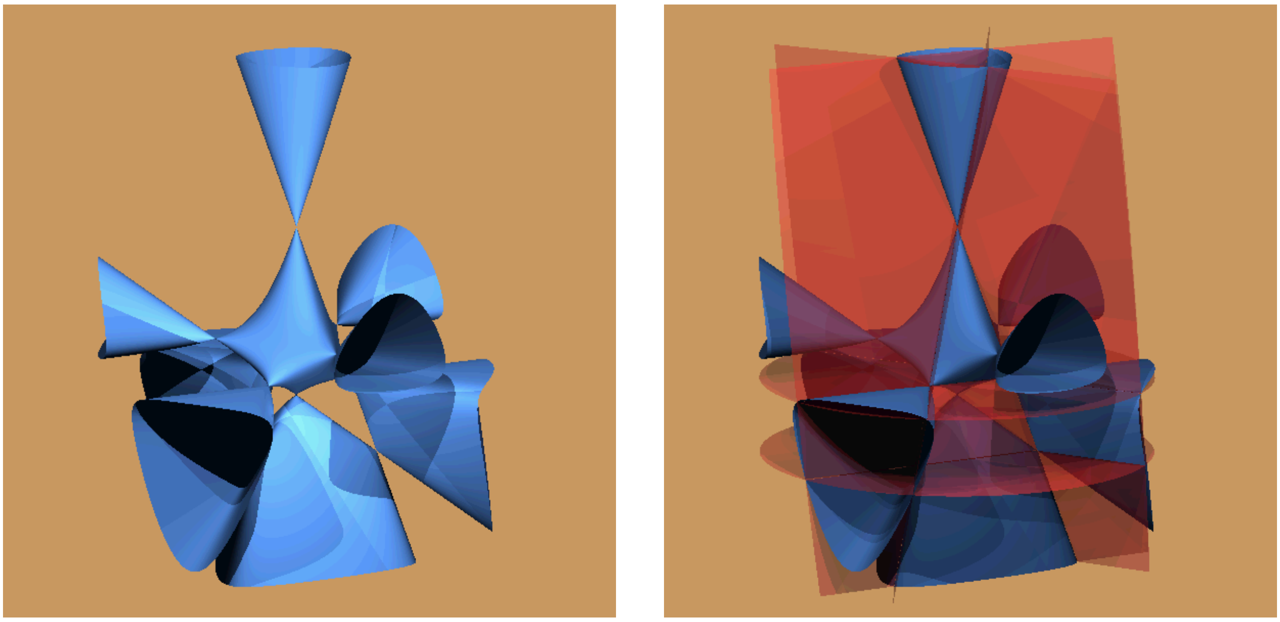
This statement becomes false over the reals (there are real cubic surfaces with 27 real lines, but they are rather special). It also becomes false if we neglect the adjective *smooth* above. An example, which is ideal to begin with, is the Cayley cubic. This is the (unique) cubic surface with four ordinary double points. So first, take a close look at the picture, and see if you can see any lines which lie on the surface. This is a beautiful example because you really can imagine them! Well, in this case, instead of 27 there are only nine lines, and six of them are just the edges of the tetrahedron whose vertices are the double points of the surface. This is easily seen in Figure 27 (a), in which we have illustrated a set of four planes which meet the surface in three lines apiece. This is the way in which the other magic number, namely 45, comes about:

There are 45 planes, called tritangent planes, each of which intersects the cubic surface in a set of three of the 27 lines.

Note than "in general" (again, there is a precise mathematical definition of what this means) a plane intersects the cubic surface in a *cubic curve*, a couple of examples of which we saw in the very first section of this talk. A set of three lines is *also* a cubic curve, one in which is the cubic has degenerated into three irreducible components.

Next, let us consider a smooth cubic surface, more precisely, the Clebsch diagonal cubic, which we have already met. Recall that this cubic surface was special in having a large, in fact the largest possible, symmetry group. From our present point of view, this surface is very special in the following way. *10 of the 45 tritangent planes of the Clebsch cubic are so-called Eckard planes, which are tritangent planes with the property that the three lines they contain all meet in a single point.* We can also see this in a nice picture, Figure 27 (b).

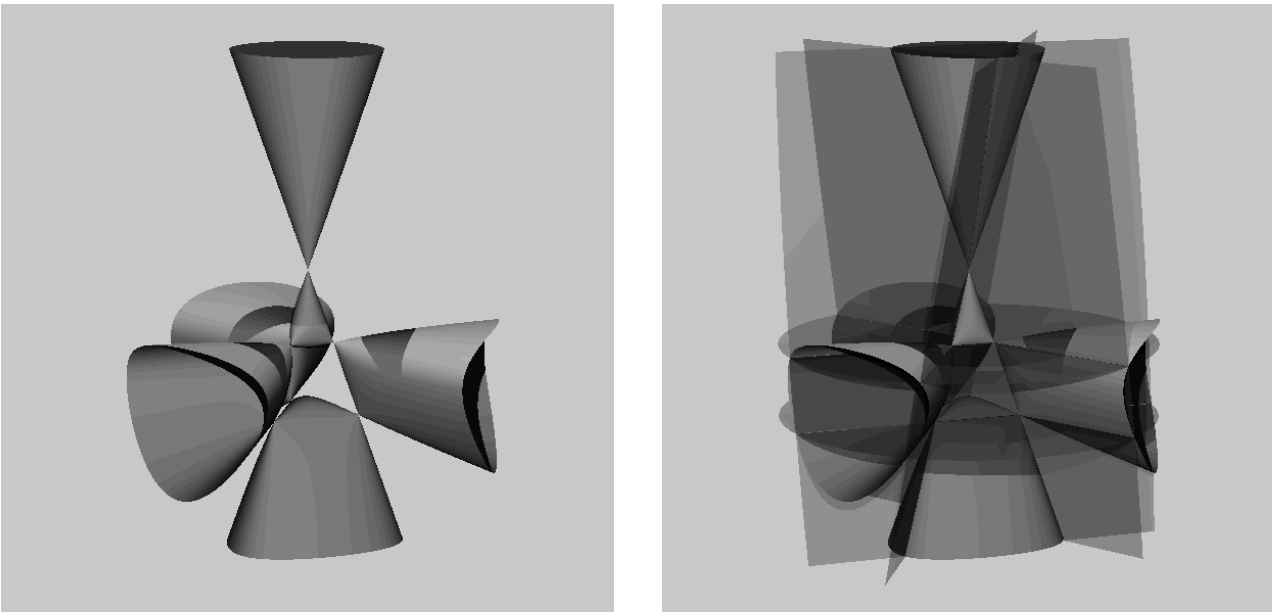
A fascinating fact about cubic surfaces concerns their *Hessian varieties.* The Hessian is obtained in a relatively simple way from the equation of the cubic, by taking the determinant of the square matrix of second derivatives of the defining polynomial, the so-called *Hessian* matrix. In the case of cubic surfaces,



(a) The Hessian of the Cayley cubic

(b) Lines on the Hessian

Figure 28: The Sylvester pentahedron and lines on the Hessian of the Cayley cubic surface



(a) The Hessian of the Clebsch cubic

(b) Lines on the Hessian

Figure 29: The Sylvester pentahedron and lines on the Hessian of the Clebsch cubic

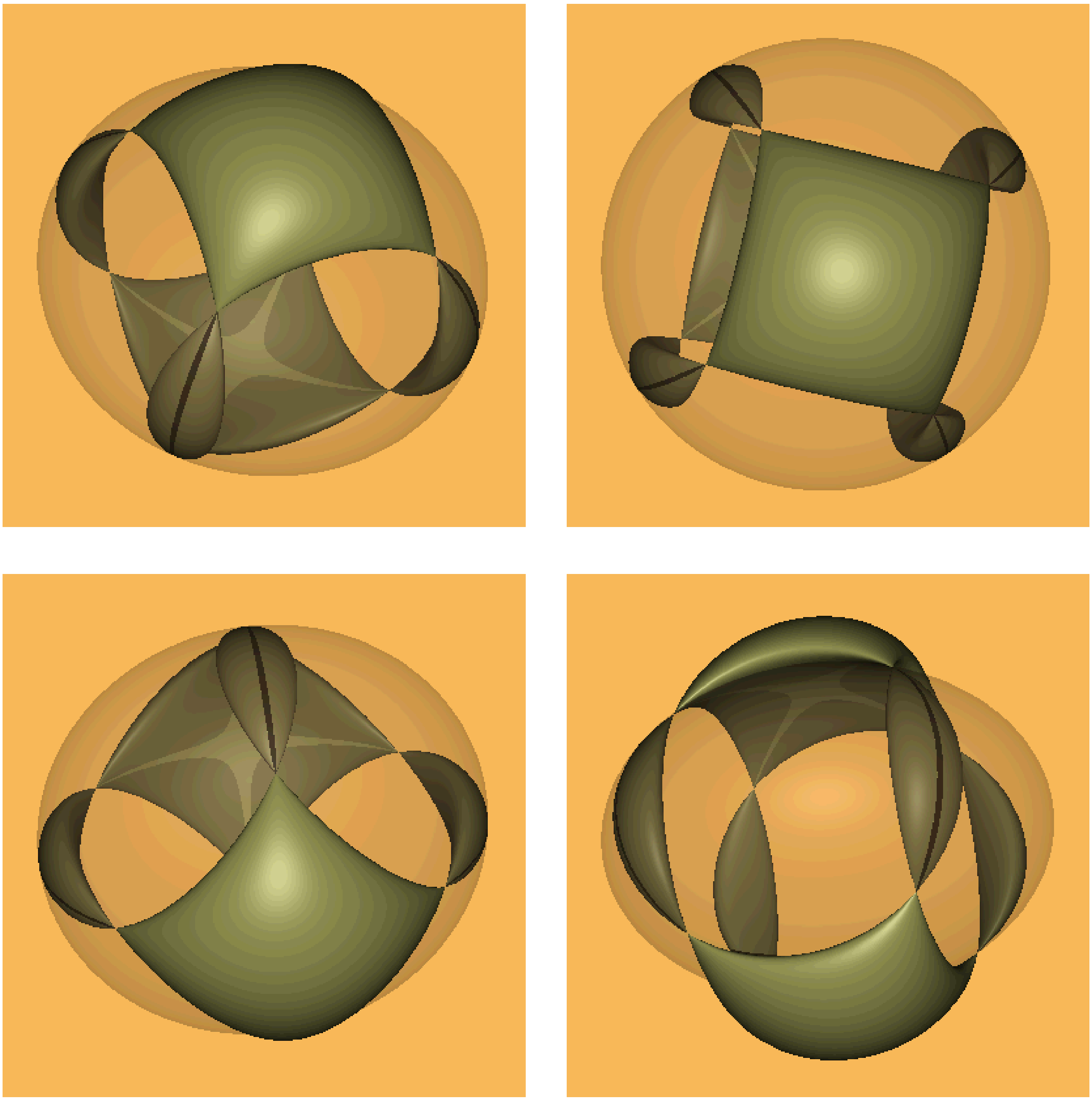


Figure 30: A quartic and quadric surface in contact

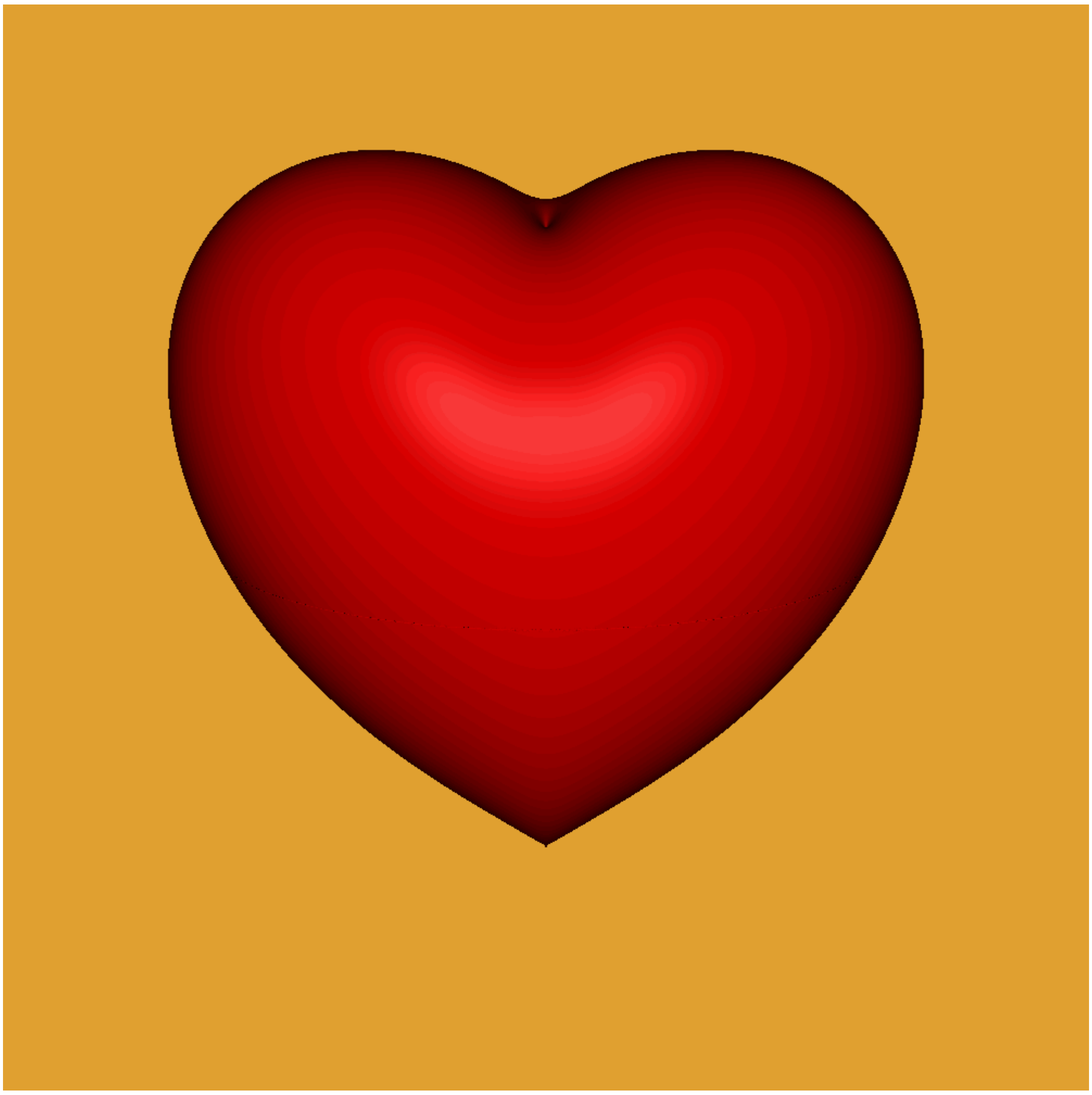
this Hessian is a *quartic surface*. In general (i.e., for a smooth cubic surface), it has 10 ordinary double points, and for every double point on the cubic, the Hessian acquires an additional node. In particular, for the Cayley cubic, the Hessian has 14 nodes instead of just 10. A nice picture of this surface is displayed in Figure 28 (a). There is a special set of five planes for a given cubic surface, a so-called *Sylvester pentahedron*, which is of use in writing down the equation. These planes are not tritangents, but it turns out they they are something similar for the Hessian. In fact, these planes meet the Hessian surface in the union of four lines each (a similar situation as the case of the tritangents for the cubics). For the Hessian of the Cayley cubic, this is depicted in Figure 28 (b). The same thing for the Hessian of the Clebsch cubic (since the Clebsch cubic is smooth, this only has ten nodes) is shown in Figure 29.

Needless to say, this is only the most simple aspect of the science of enumerative geometry. Much more interesting problems concern, instead of lines, curves of some degree. But these problems are much more difficult to display, for the simple reason that the equations involved get increasingly complex and difficult to derive. Another typical kind of question regards the notion of *tangency* of given objects. For example, how many lines are doubly tangent to a given plane curve? This question is for example extremely interesting for the famous *Klein curve*, a quartic curve in the plane, with particularly interesting properties (through which it is related to the problem of the solution of certain algebraic equations of degree seven). This curve (as does any quartic curve in the plane) has 28 bitangents, and the fact that 28 is $27 + 1$ is no coincidence: these 28 lines are in fact closely related to the 27 lines on a cubic surface. We present an image, produced by Duco v. Straten, of a quadric surface and a quartic surface which have a high degree of contact. This is shown in Figure 30.

5 Conclusion

I hope that I have convinced the audience of the beauty and the interest in considering algebraic surfaces as pieces of art. And these pieces of art are really creations of nature, of the natural world of mathematical objects, which mathematicians just endeavor to discover.

In finishing, I would like to mention that also mathematicians have a heart. Being an algebraic geometer, of course this is an algebraic surface for me:



The equation for this surface is:

$$\left(2x^2+y^2+z^2-1\right)^3-\frac{1}{10}x^2z^3-y^2z^3=0.$$

I got the equation for this surface from the web gallery of Tore Nordstrand, located at `www.uib.no/People/nfytn/math` thanks to him for this. His picture of the surface was made with a different ray tracing program, the picture above has been made with VORT.