Solving Polynomials by Iteration

An Aesthetic Approach

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Abstract. To solve a polynomial you need a means of breaking its symmetry. In the case of a generic equation of degree n, this is the symmetric group $\mathcal{S}_n$. If we extract the square root of the polynomial's discriminant, the group reduces to the alternating group $\mathcal{A}_n$. An iterative algorithm for solving the equation has two ingredients:

- **Geometric:** a complex projective space S upon which the polynomial's symmetry group acts faithfully
- **Dynamical:** a mapping of S that respects the group action—sends group-orbits to group-orbits.

This paper discusses these two aspects in the cases of the fifth and sixth degree equations. Motivating the project is a desire to develop an algorithm wiht especially elegant qualities.

0 Preliminary Background

Polynomials

One of the basic objects of mathematical study is the polynomial in one variable: an expression made up of arithmetic combinations of numbers (coefficients) and an unknown quantity (the variable). For instance, the expression

$$x^2 - 3\,x + 2$$

is a polynomial of degree two (the degree is the highest apparent power of the variable x).

Mathematicians have produced a long history of developing methods for solving *polynomial equations*: finding numbers that make the polynomial take on the value zero when they replace the variable. In the example above, the numbers 1 and 2 solve the equation

$$x^2 - 3\,x + 2 = 0.$$

A polynomial has as many solutions—also called *roots*—as its degree, provided that you count them properly and allow yourself to use complex numbers.

There is a correspondence between a polynomial and its roots—the roots determine the polynomial. If you know the roots, then, essentially, you know the polynomial. So, we can think of a polynomial in a geometric way. In our example, the two points $(1, 2)$ and $(2, 1)$ in a 2-dimensional space of ordered pairs of numbers correspond to the *same* polynomial. We can switch the coordinates of either of these points and get a different point but the same polynomial. In this way, every polynomial has symmetry; if you increase the degree, the dimension of the space of roots and the amount of symmetry also increase..

Projective Space

The set of points described by an *ordered* collection of two complex numbers (x, y) is called $\mathbf{C}^2$ or 2-dimensional complex space. By treating the lines through the point $(0, 0)$ *as points*, this space projects to a 1-dimensional space $\mathbf{CP}^1$ (called *complex projective* 1*-space* or the *complex projective line*). The same structure holds in a complex space of any dimension. Thus, the 3-dimensional $\mathbf{C}^3$ projects to the 2-dimensional space $\mathbf{CP}^2$, etc.

If we restrict our attention to points whose coordinates are *real* numbers, we get *real projective spaces* $\mathbf{RP}^1$, $\mathbf{RP}^2$, etc. Since it takes two real numbers to specify a complex number (for example, $1 + 2\,i$ where i is a square root of -1), a complex space has two times the number of "real dimensions" as the corresponding real space. So, the complex space $\mathbf{CP}^3$ has six real dimensions (three complex dimensions) whereas, the real space $\mathbf{RP}^3$ has three real dimensions.

Symmetric and Alternating Groups

Given a number n of things, there are $n! = n\cdot(n-1)\cdot\ldots\cdot 3\cdot 2$ ways of arranging them. A way of getting from one arrangement to another is a *permutation* of the objects. The set of all permutations of n things forms an object with algebraic structure called a group—specifically, the *symmetric group* $\mathcal{S}_n$. A polynomial of degree n typically has $\mathcal{S}_n$ symmetry—the basic idea is that you can permute the roots in $n!$ different ways without changing the polynomial.

The simplest permutation is to exchange two things and leave the other things alone. We can express every permutation as a succession of such *transpositions.* The permutations that decompose into an even number of transpositions also form a group—the alternating group $\mathcal{A}_n$. The number of permutations in $\mathcal{A}_n$ is half the number in $\mathcal{S}_n$.

Group Actions

When you have a set of objects S that you can move around based on the structure of a group, you are using a *group action.* In the case of solutions to polynomials of degree five, you can move the points in $\mathbf{C}^5$ (the set S in this case) that correspond to the roots by permuting their coordinates. For instance, *transform* the point $(1,2,3,4,5)$ into $(2,1,5,3,4)$ by exchanging the first two coordinates and "cycling" the third, fourth, and fifth coordinates. We say that you are "acting on" $\mathbf{C}^5$ with the symmetric group $\mathcal{S}_5$. If you use only the *even* permutations, you are acting on $\mathbf{C}^5$ with the alternating group $\mathcal{A}_5$.

The *orbit* under a group action of an element in S is the set of objects in S to which the element moves when you transform it according to *all* of the group elements. For example, under the symmetric group $\mathcal{S}_3$, the orbit of the point $(1,2,3)$ is

$$(1,2,3),\ (3,1,2),\ (2,3,1),\ (2,1,3),\ (3,2,1),\ (1,3,2).$$

Finally. a group action is *faithful* when no two elements of the group move the objects in the same way. For example, the permutation group $\mathcal{S}_n$ is faithful.

Maps

An operation that takes each point in a space A and associates it with a point in another space B is called a *mapping* (or *map*) from A to B. In this discussion, our interest is in maps from a space to itself (when $B = A$). To illustrate, take a point (x,y) in a 2-dimensional space and "send it to" the point each of whose coordinates are the squares of the original:

$$(x,y) \longrightarrow (x^2,y^2).$$

Here, the arrow means "goes to" so that

$$\begin{aligned}(2,3) &\longrightarrow (4,9)\\ (-1,i) &\longrightarrow (1,-1)\\ (1+i,2-i) &\longrightarrow (2\,i,3-4\,i).\end{aligned}$$

This sort of map is a *dynamical system*, meaning that you can *iterate* its behavior—apply it repeatedly. For instance,

$$(2,3) \longrightarrow (4,9) \longrightarrow (16,81) \longrightarrow \cdots.$$

1 Introduction—Polynomials, Symmetry, and Dynamics

When n is less than 5, the symmetric groups $\mathcal{S}_n$ act faithfully on the complex projective line $\mathbf{CP}^1$—this space has the structure of a sphere. Corresponding to each action is a *map* whose dynamics provides for an algorithmic solution to a given nth-degree equation. By 'dynamics' we mean the process of applying a map repeatedly (iteratively) to points. For instance, Newton's method provides a direct iterative solution to quadratic polynomials, but, due to a lack of symmetry, not to higher degree equations. My interests here are the geometric and dynamical properties of complex projective maps rather than numerical estimates.

The search for elegant complex geometry and dynamics continues into degree five where $\mathcal{A}_5$ is the appropriate group, since $\mathcal{S}_5$ fails to act on the sphere. This reduction in a polynomial's symmetry requires the determination of the square root of a certain number associated with a given polynomial: the *discriminant*. Such root-taking is itself the result of an iteration, namely, Newton's method. At the core of the Doyle-McMullen algorithm is a map with icosahedral symmetry. [Doyle and McMullen 1989] Their solution to the quintic takes place in three iterative steps each of which involves iteration in one complex dimension.

An alternative approach is to work with the three-dimensional action of $\mathcal{S}_5$ that derives from the group of permutations of five variables. (Section 2) The present paper describes maps that can produce quintic solutions that run as a single iteration in three dimensions.

Pressing on to the sixth-degree leads to the two-dimensional $\mathcal{A}_6$ action of the Valentiner group $\mathcal{V}$. (Section 3) Here, the problem shifts to one of finding a $\mathcal{V}$-symmetric mapping of $\mathbf{CP}^2$ from whose attractor one calculates a given sextic's root. Providing the overall framework is the 2-dimensional $\mathcal{A}_6$ analogue of the icosahedron.

For a detailed treatment of the geometry and dynamics involved here as well as how to use both in developing a solution-procedure to the quintic and sextic see [Crass 2000] and [Crass 1999].

2 The Quintic—$\mathcal{S}_5$ Acts in Three Dimensions

The permutation action of $\mathcal{S}_5$ on $\mathbf{C}^5$ does not change the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 0.$$

Therefore, it preserves the hyperplane $\mathcal{H}$—sends $\mathcal{H}$ to itself—consisting of points whose coordinates satisfy this equation, that is, add up to zero:

$$\mathcal{H} = \{(x_1, x_2, x_3, x_4, x_5) \text{ such that } x_1 + x_2 + x_3 + x_4 + x_5 = 0\}.$$

This is a 4-dimensional space which projects to 3-dimensional complex projective space $\mathbf{CP}^3$ so that the action of $\mathcal{S}_5$ on $\mathcal{H}$ creates an action of $\mathcal{S}_5$ on $\mathbf{CP}^3$. Let $\mathcal{G}_{120}$ denote the *group* of 120 transformations on $\mathbf{CP}^3$ corresponding to the permutations of $\mathcal{S}_5$.

For many purposes, the most perspicuous geometric description of the $\mathcal{G}_{120}$ action employs five coordinates that sum to zero. For example, the point $(1, 2, 3, 4, -10)$ belongs to $\mathcal{H}$. To refer to the corresponding point in projective space, I will use square brackets: $[1, 2, 3, 4, -10]$.

2.1 Invariant Polynomials

If you permute the coordinates of the point

$$(x_1, x_2, x_3, x_4, x_5),$$

the expression (also called a *polynomial in five variables*)

$$F_2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2$$

does not change. Likewise, the expressions

$$\begin{aligned} F_3 &= x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 \\ F_4 &= x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4 \\ F_5 &= x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 \end{aligned}$$

are $\mathcal{S}_5$*-invariant*. A fundamental fact is that every polynomial that is invariant under the $\mathcal{S}_5$ action on $\mathcal{H}$—permutation of its variables—has a unique expression in terms of these four polynomials.

2.2 Quadric Surface

The degree-2 invariant defines an $\mathcal{S}_5$-invariant set in $\mathbf{CP}^3$: the points whose coordinates satisfy the equation

$$F_2 = 0.$$

(Notice that a significant difference between real and complex spaces appears here: although there are infinitely many points in $\mathcal{H}$ that satisfy $F_2 = 0$, the only such point with real number coordinates is $(0, 0, 0, 0, 0)$.) This *quadric surface* $\mathcal{Q}$ consists of two families of complex projective lines $\mathcal{L}_a$, $\mathcal{L}_b$. (Note that $\mathcal{Q}$ is a *complex surface*—it has two complex dimensions.) Distinct lines in the family $\mathcal{L}_a$ (or $\mathcal{L}_b$) do not intersect while, at each point on $\mathcal{Q}$, exactly one a-line and one b-line intersect.

Furthermore, as a set, each *family* of lines—called a *ruling* on $\mathcal{Q}$—has the geometry of the icosahedron. In addition, a transformation in $\mathcal{G}_{120}$ sends lines in one ruling to either another line in the same ruling or a line in the other ruling. The set of transformations of the former type form a *subgroup* $\mathcal{G}_{60}$ of $\mathcal{G}_{120}$ that amounts to the rotational symmetries of the icosahedron.

2.3 Special Orbits

The 3-dimensional $\mathcal{S}_5$ action comes in both real and complex versions. This means that $\mathcal{G}_{120}$ acts on $\mathcal{R}$—the real projective 3-space of points whose coordinates are real numbers that sum to zero. Table 1 in Appendix A enumerates some special orbits contained in $\mathcal{R}$ while Table 2 describes elements of

$\mathcal{Q}$ that are fixed by some members of $\mathcal{G}_{120}$. For ease of expression, I will refer to special points (or lines, planes, etc.) in terms of the orbit size: "20-points" (10-lines, 5-planes).

Corresponding to each special point $a = (a_1, a_2, a_3, a_4, a_5)$ is the plane made of points whose coordinates satisfy the equation

$$a_1\, x_1 + a_2\, x_2 + a_3\, x_3 + a_4\, x_4 + a_5\, x_5 = 0.$$

In the case of the 10-points

$$[1, -1, 0, 0, 0],\ [1, 0, -1, 0, 0],\ \ldots,\ [0, 0, 1, 0, -1],\ [0, 0, 0, 1, -1]$$

there are 10-planes determined by the equations

$$x_1 = x_2,\ x_1 = x_3,\ \ldots,\ x_3 = x_5,\ x_4 = x_5.$$

Another noteworthy orbit is that of the five *coordinate planes* consisting of points one of whose coordinates is zero. The intersection of each such coordinate plane with the quadric $\mathcal{Q}$ produces a 1-dimensional set—a sphere—with the geometry of the octahedron. Some data for special two-dimensional orbits appear in Table 3.

Finally, a number of special lines appear as intersections of the 5-planes and 10-planes. Table 4 summarizes the situation.

2.4 Configurations

Some of the geometry that will have dynamical significance shows up in various collections of lines. First, the 10-lines whose points have three equal coordinates form a *complete graph* on the 5-points. Figure 1 illustrates this in two ways. The pentagon-pentagram figure displays a 5-fold symmetry while the double pyramid exhibits the 6-fold symmetry of a single 10-line—represented by the polar axis.

Within each of the icosahedral rulings on $\mathcal{Q}$ there are three special line-orbits. These correspond to the 12 vertices, 20 face-centers, and 30 edge-midpoints of the icosahedron. Intersections of lines between rulings give special point structures.

- Two 12-line $\mathcal{G}_{60}$-orbits form six "quadrilaterals" at 24-points.
- Two 20-line $\mathcal{G}_{60}$-orbits form ten quadrilaterals at two pairs of 20-points. (See Figure 2.)
- Two 30-line $\mathcal{G}_{60}$-orbits orbits form 15 quadrilaterals at two pairs of 30-points.

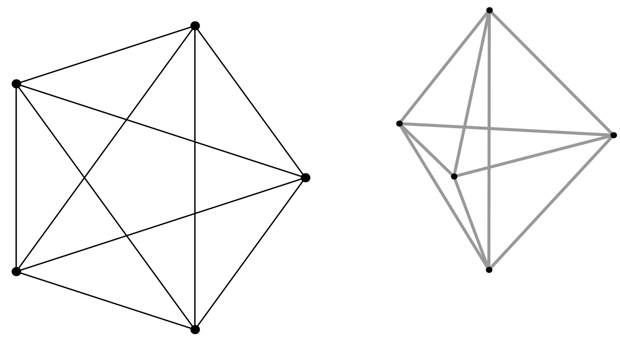

Fig. 1. Configuration of 10-lines and 5-points

2.5 Dynamical Terminology.

The *trajectory* of a point x under a map f is the set of points obtained by "applying" f iteratively to x. A point p is *periodic* if its trajectory contains p more than once. A periodic point a in a space X is *attracting* when the trajectory of every point near a gets arbitrarily close to a. The *basin of attraction* of a is the set of all points attracted to a. Also, the *attractor* of f is the set of all attracting points.

2.6 Equivariant Maps

The primary tool to be used in solving the general quintic is a map that associates points in $\mathbf{CP}^3$ with points in $\mathbf{CP}^3$ in a way that respects the action of the group of transformations $\mathcal{G}_{120}$. We want to find a $\mathcal{G}_{120}$-*equivariant* map (or simply $\mathcal{G}_{120}$-*equivariant*) with elegant geometry and *reliable dynamics*; this means that its attractor

1) is a *single* orbit under $\mathcal{G}_{120}$
2) has a corresponding basin that "fills up" $\mathbf{CP}^3$.

2.7 Basic Maps

The four maps indicated by

$$\begin{aligned}
f_1 &: [x_1, x_2, x_3, x_4, x_5] \longrightarrow [x_1, x_2, x_3, x_4, x_5] \\
f_2 &: [x_1, x_2, x_3, x_4, x_5] \longrightarrow [x_1^2, x_2^2, x_3^2, x_4^2, x_5^2] \\
f_3 &: [x_1, x_2, x_3, x_4, x_5] \longrightarrow [x_1^3, x_2^3, x_3^3, x_4^3, x_5^3] \\
f_4 &: [x_1, x_2, x_3, x_4, x_5] \longrightarrow [x_1^4, x_2^4, x_3^4, x_4^4, x_5^4]
\end{aligned}$$

are $\mathcal{G}_{120}$-equivariant. Moreover, by combining these with the invariants

$$F_2,\ F_3,\ F_4,\ F_5,$$

we can produce all $\mathcal{G}_{120}$-symmetric maps.

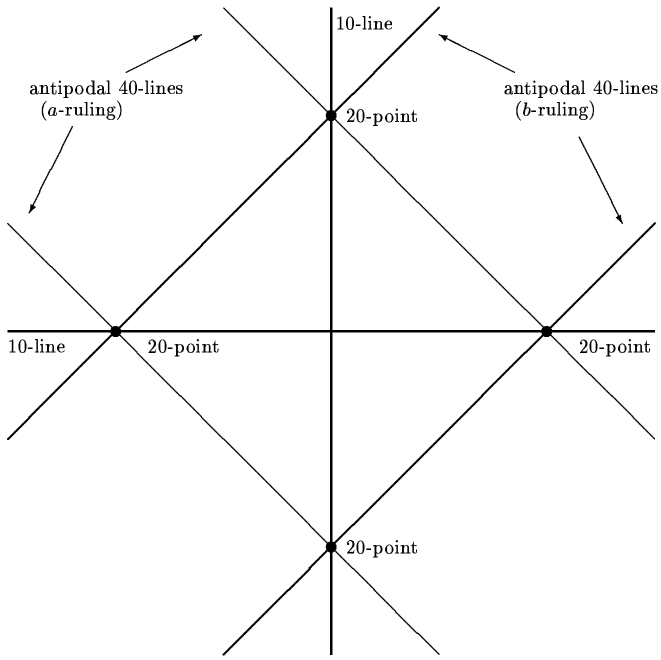

Fig. 2. Configuration of 40-lines and 20-points on $\mathcal{Q}$. At a 20-point there are two 40-lines—one in each ruling on the quadric. This pair of lines is the intersection of $\mathcal{Q}$ with the tangent plane to $\mathcal{Q}$ at the respective 20-point. Also indicated are the 10-lines determined by a pair of antipodal 20-points.

2.8 Families of Equivariants

The $\mathcal{G}_{120}$-equivariants form a *module* over the $\mathcal{G}_{120}$-invariants. This means that for an invariant F_ℓ and equivariant g_m of degrees ℓ and m, the product

$$F_\ell \cdot g_m$$

is an equivariant of degree $\ell + m$. When looking for a map in a certain degree with special geometric or dynamical properties, my approach is to express the entire family of equivariants for that degree and, by manipulation of parameters, locate a subfamily and eventually a single map with interesting behavior.

2.9 Quadric-preserving Maps

The rich geometry of the quadric $\mathcal{Q}$ provides an intriguing setting for dynamical exploration. Are there $\mathcal{S}_5$-symmetric maps that send $\mathcal{Q}$ to itself? If so, how do they behave on and off $\mathcal{Q}$? I will describe two species of such maps: one associated with the icosahedron and the other with the octahedron.

Maps that Preserve Icosahedral Rulings Were a $\mathcal{G}_{120}$-equivariant to preserve the rulings on $\mathcal{Q}$, its restriction to either ruling $\mathcal{L}_a$ or $\mathcal{L}_b$ would express itself in terms of the basic equivariants under the one-dimensional icosahedral action. Maps of this kind occur in degrees 11, 19, and 29. [Doyle and McMullen 1989, p. 166] Consequently, the 20-parameter family of 11-maps comes under scrutiny:

$$\begin{aligned} f_{11} = {} & (\alpha_1 F_2^5 + \alpha_2 F_2^2 F_3^2 + \alpha_3 F_2^3 F_4 + \alpha_4 F_3^2 F_4 + \alpha_5 F_2 F_4^2 + \alpha_6 F_2 F_3 F_5 + \alpha_7 F_5^2)\, f_1 \\ & + (\alpha_8 F_2^3 F_3 + \alpha_9 F_3^3 + \alpha_{10} F_2 F_3 F_4 + \alpha_{11} F_2^2 F_5 + \alpha_{12} F_4 F_5) f_2 \\ & + (\alpha_{13} F_2^4 + \alpha_{14} F_2 F_3^2 + \alpha_{15} F_2^2 F_4 + \alpha_{16} F_4^2 + \alpha_{17} F_3 F_5)\, f_3 \\ & + (\alpha_{18} F_2^2 F_3 + \alpha_{19} F_3 F_4 + \alpha_{20} F_2 F_5)\, f_4. \end{aligned}$$

Using the parameters α_1 through α_{20} to satisfy the demands of the 1-dimensional 11-map with icosahedral symmetry, we obtain a 13-parameter family of ruling-preserving maps

$$\begin{aligned} g_{11} = {} & 4\left(16\,\alpha_1 F_2^5 + 16\,\alpha_2 F_2^2 F_3^2 + 16\,\alpha_3 F_2^3 F_4 + 67\, F_3^2 F_4\right. \\ & + \left. 16\,\alpha_5 F_2 F_4^2 + 16\,\alpha_6 F_2 F_3 F_5 + 45\, F_5^2\right) f_1 \\ & + 4\left(16\,\alpha_8 F_2^3 F_3 + 16\, F_3^3 + 16\,\alpha_{10} F_2 F_3 F_4 + 16\,\alpha_{11} F_2^2 F_5 - 135\, F_4 F_5\right) f_2 \\ & + \left(64\,\alpha_{13} F_2^4 + 64\,\alpha_{14} F_2 F_3^2 + 64\,\alpha_{15} F_2^2 F_4 + 405\, F_4^2 - 720\, F_3 F_5\right) f_3 \\ & + 4\left(16\,\alpha_{18} F_2^2 F_3 - 225\, F_3 F_4 + 16\,\alpha_{20} F_2 F_5\right) f_4. \end{aligned}$$

Restricted to a ruling, the dynamics of each g_{11} is well-understood. The 20-lines (under $\mathcal{G}_{120}$) are exchanged in pairs. (Recall that 20-lines in $\mathcal{Q}$ are dodecahedral vertices in $\mathcal{L}_a$ or $\mathcal{L}_b$.) Moreover, almost every line in the ruling belongs to the basin of one of the ten antipodal pairs of the superattracting set of 20-lines. (See Figure 5 in Appendix B.) Thus, for almost every point x on $\mathcal{Q}$, there is an "antipodal" pair of intersections *between* 20-lines in different rulings which the trajectory of x approaches.

As a result, the global behavior of each g_{11} depends on its dynamics off $\mathcal{Q}$. Should the quadric attract or repel? If Q were attracting, then the 400 intersections of 20-lines in different rulings would attract in all directions. One way to arrange for this is to force these points to be attracting in the off-quadric direction. However, this situation does not conform to the model of *reliable* dynamics. The attractor would not be a single $\mathcal{G}_{120}$-orbit of points, though it might be the set of intersections of a single line-orbit. I have not explored the case of a repelling quadric. Such a situation would arise if intersections of 20-lines were attracting in the two quadric directions but repelling in the off-quadric direction.

An Octahedral Map Since the orbit of the five coordinate planes has fundamental geometric significance, a map that preserves these sets might exhibit interesting dynamics. Arranging for this spends four of the twenty parameters of the family f_{11}.

The intersection of a 5-plane and the quadric $\mathcal{Q}$ is a conic (sphere) with the $\mathcal{S}_4$ symmetry of the octahedron. One of the special equivariants for the octahedral action on $\mathbf{CP}^1$ is a degree-5 map that attracts almost every point to the eight face-centers— vertices of the dual cube. Geometrically, the map stretches each face F of the cube symmetrically over the five faces in the complement of the face opposite F. As a face stretches, it makes a half-turn so that the vertices and edges land on their antipodes. (See Figure 6 in Appendix B.) Under $\mathcal{G}_{120}$, antipodal pairs of octahedral face-centers are pairs of 20-points.

The idea is to find a reliable map that sends $\mathcal{Q}$ to itself and behaves like the special 5-map on each of the octahedral conics. Such a map would attract points on $\mathcal{Q}$ to the 20-points. We also want the 20-points to be attracting off $\mathcal{Q}$. In degree five there are too few parameters for the purpose. However, the 11-maps provide enough freedom to arrange for elegant geometry.

Each of the 10-lines two of whose coordinates are zero contains a pair of antipodal 20-points. A map that

1) preserves these lines,
2) attracts almost every point on the line to the 20-points, and
3) superattracts in the directions *off* the line

would act as a "superattracting pipe" to the quadric. Expenditure of the remaining parameters purchases a map h_{11} with these properties.

It happens that h_{11} also preserves $\mathcal{R}$—the $\mathcal{S}_5$-symmetric space of points whose coordinates are real numbers—as well as the 2-dimensional intersections of $\mathcal{R}$ with the five coordinate planes and the ten planes whose points have two equal coordinates. In the former case there are four intersections of the 2-dimensional space with the superattracting 10-lines while in the latter there is a single such intersection. Each such intersection, is a real projective *line* as well as an "equatorial slice" of the associated complex projective line—a sphere. When restricted to such a slice, h_{11} acts chaotically, meaning that the trajectory of most of the circle's points gets arbitrarily close to every point on the circle. A basin portrait for the 5-plane reveals no basins other than those of the four chaotically attracting 10-lines. (See Figure 9.) The dynamics on the "real part" of the 10-plane shows, in addition to the chaotic line-attractor, three additional basins at 30-points. (See Figure 10.) A 30-point belongs to a 10-line, which intersects the 10-plane transversely. Thus, near a 30-point, but off the 10-plane, there is only the "pipe-basin" of the 20-points. Hence, the basins of a 30-point are strictly 2-dimensional.

2.10 A Special Map in Degree Six

In the configuration of 10-lines each 5-point lies at the intersection of four lines. (See Section 2.4.) Moreover, these are the only intersections of 10-lines. To take advantage of this structure, a map could have superattracting pipes along the 10-lines and, thereby, have 3-dimensional basins of attraction at the 5-points.

The family of 6-maps has six free parameters. I takes four parameters to obtain maps for which the 10-lines are superattracting in the "off-line" directions. For the remaining two, we get a map f_6 whose restriction to a 10-line is

$$z \longrightarrow z^4$$

when expressed in coordinates where the 5-points on the respective 10-line are 0 and ∞. Recalling that a 10-line is a sphere, such a map attracts all points in the northern hemisphere to the 5-point at ∞ (the north pole) and attracts all points in the southern hemisphere to the 5-point at 0 (the south pole). The equatorial circle—a real projective line—maps to itself chaotically.

Of necessity, f_6 preserves each $\mathcal{S}_3$-symmetric 10-plane whose points have two equal coordinates. To be specific, consider such a 10-plane $\mathcal{L}$ whose points we can describe by

$$[x, y, z, -\frac{1}{2}(x+y+z), -\frac{1}{2}(x+y+z)].$$

The 10-point $[2, 2, 2, -3, -3]$ and 5-points

$$[-4, 1, 1, 1, 1],\ [1, -4, 1, 1, 1],\ [1, 1, -4, 1, 1]$$

form $\mathcal{S}_3$-orbits on $\mathcal{L}$. Furthermore, f_6 preserves $\mathcal{R}$—the $\mathcal{S}_5$-symmetric $\mathbf{RP}^3$. We can get a picture of the map's *restricted dynamics* by plotting basins of attraction on the $\mathbf{RP}^2$ intersection of $\mathcal{L}$ and $\mathcal{R}$. (See Figures 11 through 14 in Appendix B.) The plot shows attraction to the 5-points and the 10-point. However, the 10-point lies on the "equator" of the 10-line whose points look like $[x, x, x, y, -3\,x - y]$. Here, f_6 *repels* in the off-plane direction. Thus, the basin of a 10-point is 2-dimensional. No other attracting sets appear.

A 15-line (such as $[x, x, y, y, -2\,(x+y)]$) contains one 5-point $[1, 1, 1, 1, -4]$, one 15-point $[1, 1, -1, -1, 0]$, and two 10-points $[2, 2, -3, -3, 2]$, $[-3, -3, 2, 2, 2]$. In coordinates where the 5-point is 0, the 15-point is ∞, and the 10-points are ± 1 the map restricts to

$$z \longrightarrow \frac{48\,z^5}{-3 - z^2 + 35\,z^4 + 17\,z^6}\,.$$

(Figures 15 and 16 display portraits.)

Another distinction for f_6 is its action on a 15-line—say $[x, -x, y, -y, 0]$—which maps to the 15-line $[x, x, y, y, -2\,(x + y)]$. In fact, this is what led me to 6-maps each of which send the 10-point (such as $[0, 0, 0, 1, -1]$) to its associated 10-point [2,2,2,-3,-3].

Finally, f_6 preserves a certain 3-dimensional real projective space that is associated with the group of transformations that fix a 5-point. This $\mathbf{RP}^3$ intersects two 10-planes in an $\mathbf{RP}^2$—which you can think of as a sphere—that has the symmetry of a double rectangular pyramid. In addition to the 5-point this $\mathbf{RP}^2$ contains three 10-points as well as the $\mathbf{RP}^1$ through two of the 10-points. Since this line is an equatorial slice through a 10-line where the map looks like

$$z \longrightarrow z^4,$$

f_6 behaves chaotically along the line while attracting points off the line. (See Figure 17 for a basin portrait.)

3 The Sextic—$\mathcal{A}_6$ Acts in Two Dimensions

3.1 Basics of $\mathcal{A}_6$ and Valentiner's Group

Inside the alternating group $\mathcal{A}_6$ are twelve versions of the alternating group $\mathcal{A}_5$. These twelve subgroups decompose into two systems of six:

1) the permutations that leave one thing unmoved
2) the permutations of the six pairs of antipodal icosahedral vertices resulting from the icosahedron's rotational symmetries.

The group $\mathcal{A}_6$ can act on these subgroups by permuting each of the two systems individually. A given $\mathcal{A}_5$ subgroup fixes itself as a set and permutes the five companion subgroups in its system according to the rotational icosahedral group's action on the five cubes found in the icosahedron. Meanwhile, the other system of six $\mathcal{A}_5$ subgroups undergo the permutations of the six pairs of antipodal vertices. Consequently, the intersection of two $\mathcal{A}_5$ subgroups in the *same* system is isomorphic to the group $\mathcal{A}_4$—the tetrahedral rotations—while two in *different* systems give a *dihedral* group $\mathcal{D}_5$—the symmetries of a double pentagonal pyramid.

In the late nineteenth century, Valentiner discovered a group—call it $\mathcal{V}$—of 360 transformations of 2-dimensional complex projective space that has the same structure as the group of permutations $\mathcal{A}_6$. To solve the sextic equation, we must find a map on $\mathbf{CP}^2$ that is symmetric with respect to $\mathcal{V}$.

3.2 Valentiner Geometry

Icosahedral Conics The $\mathcal{A}_5$ subgroups of $\mathcal{A}_6$ correspond to subgroups of $\mathcal{V}$. Each of the $\mathcal{A}_5$ subgroups preserve a respective 1-dimensional *conic*—a sphere—which thereby has the geometry of the icosahedron. The group $\mathcal{V}$ permutes these two sets of six icosahedra in the same way as $\mathcal{A}_6$ permutes its two systems of $\mathcal{A}_5$ subgroups.

Special Orbits Some of the special icosahedral points on a conic occur at its intersections with the other 11 conics. There are two cases.

- Two conics *in the same* system intersect in four tetrahedral points; this gives the 20 icosahedral face-centers on a given conic. The overall result is a 60 point $\mathcal{V}$-orbit for each system of conics.
- Two conics *in different* systems intersect in two points. This gives six pairs of antipodal icosahedral vertices on each conic. These total to a $\mathcal{V}$-orbit consisting of $72 = 6 \cdot 12$ points. Figure 3 illustrates the situation.

As for other special orbits, each of the 45 transpositions in $\mathcal{A}_6$ corresponds to a transformation T in $\mathcal{V}$ that fixes every point on a line associated with T. In addition, T fixes a point that is *not* on its associated line. These give $\mathcal{V}$-orbits of 45 lines and 45 points. An equivariant map typically preserves each of these lines and points; however, as we will see (Section 3.4), something quite different can occur. The typical points *on* a 45-line lie in four-point orbits and, overall, provide $\mathcal{V}$-orbits of size 180. Other special orbits occur at the intersections of the 45-lines:

- 36-points on five of the 45-lines
- 45-points on four of the 45-lines
- 60-points on three of the 45-lines.

The 72-points have the distinction of being the only special $\mathcal{V}$-orbit that do not belong to the 45-lines.

An Additional Symmetry The one-dimensional icosahedral group $\mathcal{G}_{60}$ acts on two sets of five tetrahedra each of which corresponds to a quadruple of face-centers on the icosahedron. However, no element of the group sends the tetrahedra of one set to those of the other. Such an exchange occurs by means of orientation-reversing transformations. Some of these are reflections through the 15 great circles of reflective icosahedral symmetry; the remaining

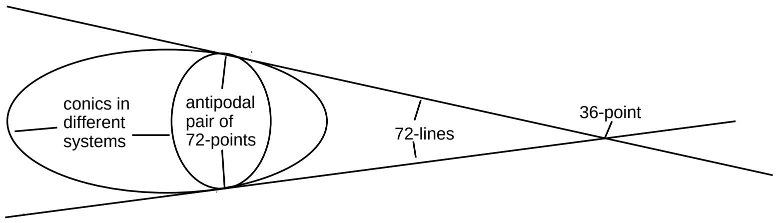

Fig. 3. The triangle of one 36-point and two 72-points.

45 are the various "odd" compositions of these 15 basic reflections—e.g., the map that sends a point to its antipode. *Extending* the orientation-preserving group $\mathcal{G}_{60}$ by such an orientation-reversing transformation produces the group of all 120 symmetries of the icosahedron.

The Valentiner analogues of the tetrahedra are the two systems of conics. There are orientation-reversing transformations of $\mathbf{CP}^2$ that exchange the systems of conics. By taking all combinations of such a transformation with the elements in the group $\mathcal{V}$, we get a new group consisting of 720 symmetries of the Valentiner structure. In analogy to the 15 great circle reflections that produce all 120 symmetries of the icosahedron, there are 36 orientation-reversing transformations that combine to make the 720 Valentiner symmetries. Each of these 36 transformations fixes every point of an associated real-projective plane. (Figure 4 illustrates a geometric construction for these planes.) These 36 planes stand in analogy to the 15 great circles—real projective lines—of icosahedral reflections. A map that respects all 720 of the Valentiner symmetries must send each of these planes to itself. This circumstance allows us to make pictures of such a map's dynamical behavior.

3.3 Invariant Polynomials and Equivariant Maps

Every polynomial that is invariant under the Valentiner group can be expressed as a combination of four *basic* invariants. We can obtain any $\mathcal{V}$-equivariant map from combining these invariants and their derivatives (in the sense of calculus). Again, the idea is to employ a palette of parameters in designing a geometrically elegant map.

3.4 The lowest degree equivariant—a case of inelegant dynamics

In degree 16 we find the map of least degree with Valentiner symmetry. This map has the property that it smashes a 45-line down to its associated 45-point. Furthermore, it "blows-up" a 45-point to its companion 45-line. This

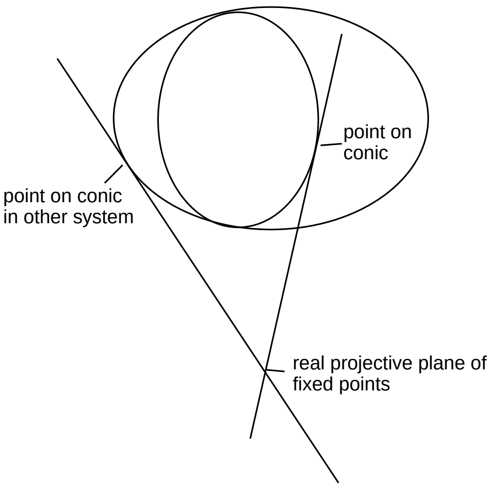

Fig. 4. A geometric interpretation of transformations that exchange conics in different systems—indicated by the pair of points.

means that the map spreads points near the 45-point over the 45-line. The basin portrait (Figure 18) fails to reveal the geometric elegance for which we seek.

3.5 A Special Icosahedral Map of Degree 19

Associated with the icosahedron is a degree-19 map that takes each of the 20 faces and stretches it around the icosahedron omitting the opposite face. When iterated, this icosahedral equivariant attracts almost any point of the icosahedron to one of the six pairs of antipodal vertices. (See Figure 19.)

In the higher dimensional case of the Valentiner group, there is a 19-map h_{19} that send each of the 12 icosahedral conics to itself. This means that *on* a conic h_{19} is the map described above, so that we understand much of its dynamics there. Recall that the vertices of the icosahedral conics make up the 72 point $\mathcal{V}$-orbit. It happens that away from the conic, these points are also attracting.

Moreover, h_{19} has the additional symmetries of the transformations that exchange the systems of conics. Therefore, it preserves each of the 36 real projective planes associated with these transformations. These spaces look very much like familiar two-dimensional planes. The map's dynamical behavior on such a plane appears in Figures 20 through 24.

3.6 A Special Dodecahedral Map of Degree 11

There is another map that preserves conics, though this one is of a kind different from that of the 19-map. In this case, we use *complex conjugation*: an operation on the complex numbers that reflects a point through the axis of real numbers in the plane of complex numbers. This is the sort of orientation-reversing process that exchanges the two sets of five tetrahedra in the icosahedron. We can also apply it to the coordinates of points in 2-dimensional space; this type of operation exchanges the two systems of conics.

For each system of conics, but not for both, there is an orientation-reversing map that preserves the six conics individually. When restricted to one of the conics, the map's geometry is to stretch each dodecahedral face onto its complement—the sphere minus the face—while fixing the vertices and edges and sending the face-center to its antipode. You can imagine pushing the face *inside* the dodecahedron spreading it out symmetrically onto the other 11 faces. This defines an 11-map expressed in complex conjugated coordinates. When iterated, the 12 fixed-points at the vertices attract almost all points on the sphere. Since two conics in one system intersect transversely at the vertices (60-points), these points are attracting in all directions. I plan to study this map and, in a future paper, give a detailed description of its behavior.

A Special Orbit Data

For ease of reference, the following tables provide descriptions of the special $\mathcal{G}_{120}$ orbits associated with the maps discussed in the text.

Table 1. Special points on the $\mathcal{S}_5$-symmetric real projective plane

Number of points	Representative point
5	$[-4, 1, 1, 1, 1]$
10	$[0, 0, 0, 1, -1]$
10	$[2, 2, 2, -3, -3]$
15	$[0, 1, 1, -1, -1]$
20	$[0, -3, 1, 1, 1]$
30	$[0, 0, 1, 1, -2]$

Table 2. Special points on the quadric $\mathcal{Q}$

Size	Representative	Remarks
20	$[0, 0, 1, \omega_3, \omega_3^2]$ $[0, 0, 1, \omega_3^2, \omega_3]$	antipodal pair of eight octahedral face-centers on intersections of $\mathcal{Q}$ and a coordinate plane $\omega_3 = e^{2\pi i/3}$
20	$[1, 1, 1, \alpha, \bar{\alpha}]$ $[1, 1, 1, \bar{\alpha}, \alpha]$	$\alpha = \frac{-3+\sqrt{15}\,i}{2}$
24	$[1, \omega_5, \omega_5^2, \omega_5^3, \omega_5^4]$	$\omega_5 = e^{2\pi i/5}$
30	$[0, 1, i, -1, -i]$ $[0, 1, -i, -1, i]$	antipodal pair of six octahedral vertices on intersections of $\mathcal{Q}$ and a coordinate plane
30	$[1, 1, \beta, \beta, -2(1+\beta)]$ $[1, 1, \bar{\beta}, \bar{\beta}, -2(1+\bar{\beta})]$	$\beta = \frac{-2+\sqrt{5}\,i}{3}$
60	$[0, 1, 1, \gamma, \overline{\gamma}]$ $[0, 1, 1, \overline{\gamma}, \gamma]$	antipodal pair of 12 octahedral edge-midpoints on intersections of $\mathcal{Q}$ and a coordinate plane $\gamma = -1 + \sqrt{2}\,i$

Table 3. Special orbits of complex projective planes

Size	Algebraic definition	Corresponding point
5	$x_1 = 0, \ \ldots, \ x_5 = 0$	$[-4, 1, 1, 1, 1], \ \ldots, \ [1, 1, 1, 1, -4]$
10	$x_1 = x_2, \ \ldots, \ x_4 = x_5$	$[1, -1, 0, 0, 0], \ \ldots, \ [0, 0, 0, 1, -1]$
10	$x_1 = -x_2, \ \ldots, \ x_4 = -x_5$	$[-3, -3, 2, 2, 2], \ \ldots, \ [2, 2, 2, -3, -3]$

Table 4. Special orbits of complex projective lines

Size	Description
10	$[x, y, -(x+y), 0, 0]$
10	$[x, x, x, y, -3\,x - y]$
15	$[x, x, y, y, -2\,(x+y)]$
15	$[x, -x, y, -y, 0]$
30	$[x, x, y, -2\,x - y, 0]$

B Gallery of Basin Portraits

The basin plots that follow are productions of the program *Dynamics* and *Dynamics 2* that ran respecctively on a Silicon Graphics Indigo-2 and a Dell Dimension XPS with a Pentium II processor. Its BA and BAS routines produced the images. (See the manuals [Nusse and Yorke 1994] and [Nusse and Yorke 1998].) Each procedure divides the screen into a grid of cells and then colors each cell according to which attracting point its trajectory approaches. If it finds no such attractor after 60 iterations, the cell is black. The BA algorithm finds the attractor whereas BAS requires the user to specify a candidate attracting set of points. Each portrait exhibits the highest resolution available—a 720×720 grid.

Maps with $\mathcal{S}_5$ Symmetry

Figure 5: The dodecahedral 11-map. Each of the ten pairs of antipodal dodecahedral vertices—black dots—is a period-2 superattractor. Their basins fill up $\mathbf{CP}^1$. (Bear in mind that points in the space of this plot correspond to lines in either ruling on the quadric surface $\mathcal{Q}$.)

Figure 6 This plot indicates the behavior of h_{11} restricted to an $\mathcal{S}_4$-symmetric conic—the intersection of a coordinate plane and the quadric $\mathcal{Q}$. The four pairs of antipodal vertices of the cube are period-2 superattracting 20-points whose basins fill up the conic.

Figures 7 and 8 These show the behavior of the octahedral map h_{11} on a 15-line and a 30-line respectively. In the former case, the superattracting points at 0 and ∞ are a pair of 30-points on $\mathcal{Q}$ that h_{11} exchanges. A pair of fixed 10-points accounts for the remaining two basins. At each of these attracting points, the map repels in at least one direction away from the line.

On the 30-line, the superattracting points at 0 and ∞ are a pair of 60-points (antipodal edge-midpoints) on an octahedral conic; the map exchanges the two points. The remaining two basins belong to a pair of 20-points on $\mathcal{R}$. As before, the map repels in at least one direction away from the line at each of these attracting points.

Figures 9 and 10 We see the restriction of h_{11} to an $\mathbf{RP}^2$ with $\mathcal{S}_4$ symmetry and an $\mathbf{RP}^2$ with $\mathcal{S}_3$ symmetry. Each case involves a chaotic attractor. In the former, the attractor consists of the four $\mathbf{RP}^1$ intersections of $\mathcal{R}$, a coordinate plane, and four of the 10-lines with two zero coordinates. The six intersections of 10-lines occur at 10-points with three zero coordinates. (In the picture, two of these intersections occur on the line at infinity.) The pictured "lines" are the images of small circles centered along the edges of the inner square. This graphical technique specifically relies on the chaotic and attracting behavior of h_{11} along each line.

In the $\mathcal{S}_3$-symmetric plane, the attracting line is the intersection of $\mathcal{R}$, a 10-plane with two zero coordinates and the 10-line at infinity—the light gray basin. The three "attracting" 30-points—they are blowing up—are the vertices of an equilateral triangle.

The remaining images illustrate the dynamics of the 6-map f_6.

Figures 11 through 14 We see the restriction to the $\mathbf{RP}^2$ determined by the intersection of $\mathcal{R}$ and a 10-plane with two equal coordinates. Since this plane is $\mathcal{S}_3$-symmetric, we select the coordinates so that the three 5-points are vertices of an equilateral triangle centered at $(0,0)$. Three of the superattracting pipes contain this triangle. Indeed, as Figure 11 shows, the map sends the circle of radius $\frac{1}{4}$ centered at $(0,0)$ nearly to this triangle. The attractor at $(0,0)$ is the 1-point orbit *in* the 10-plane—overall, a 10-point (of the type $[-3,-3,2,2,2]$). In the direction away from the plane, f_6 repels at this site along a superattracting pipe. The three "spokes" at basin boundaries are pieces of 15-lines each of which passes through a secondary basin that contains a point that maps to the central 10-point.

Figure 12 This shows h_{11}'s critical set—where the map folds the plane over—superimposed on the blurry basin portrait. The critical contour is a *Mathematica* plot. The curve crosses itself at the 5-points. All but six critical points appear to belong to the basin of either a 5-point or the central 10-point. The six exceptions lie on the 15-lines at basin boundaries. If this is so, then there is no other attracting site.

Figures 15 and 16 We see the map restricted to a 15-line that maps to itself. The coordinates of this image place the single 5-point at 0 and the two fixed superattracting 10-points at ± 1. At the latter points, the map repels in all directions off the line. Figure 16 approximately shows the boxed region.

Figure 17 The space is the $\mathbf{RP}^2$ intersection of an $\mathcal{S}_4$-invariant $\mathbf{RP}^3$ and a 10-plane with two equal coordinates. The $\mathbf{RP}^1$ intersection of the $\mathbf{RP}^2$ and the associated 10-line with three equal coordinates is the central vertical axis. By plotting the trajectory of one of its generic points, this line reveals itself as a chaotic attractor; the plot shows roughly $20,000$ iterates. The map attracts at $(1,0)$ and $(-1,0)$—a 5-point and 10-point respectively.

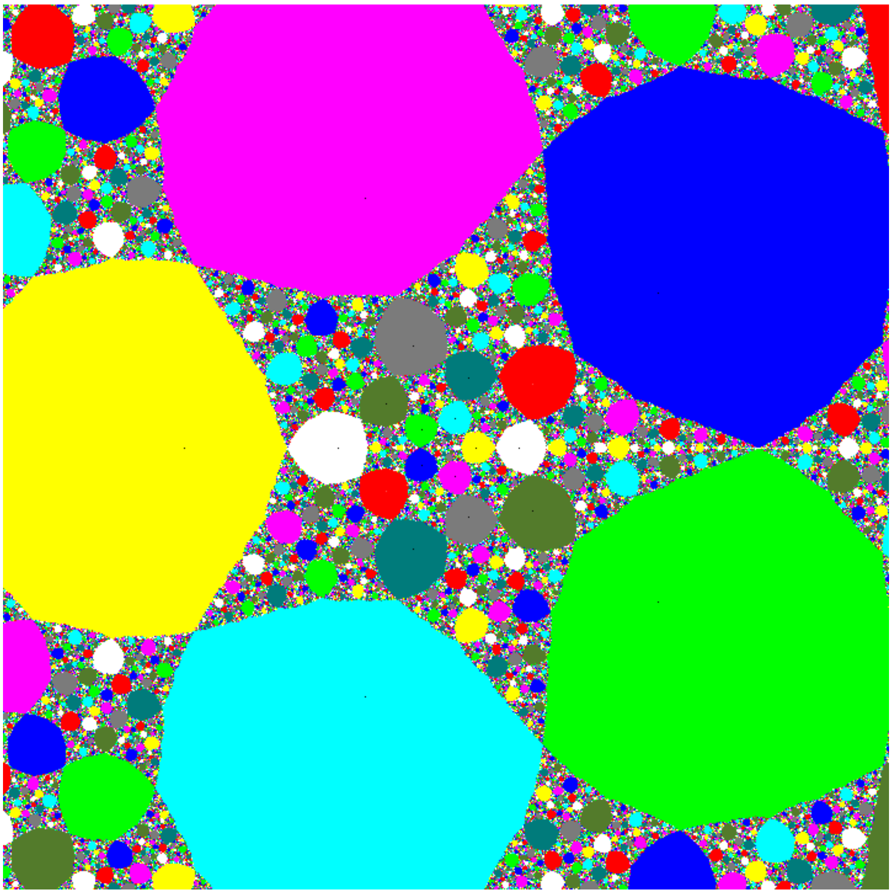

Fig. 5. Dynamics of a ruling-preserving 11-map on each of the quadric's rulings

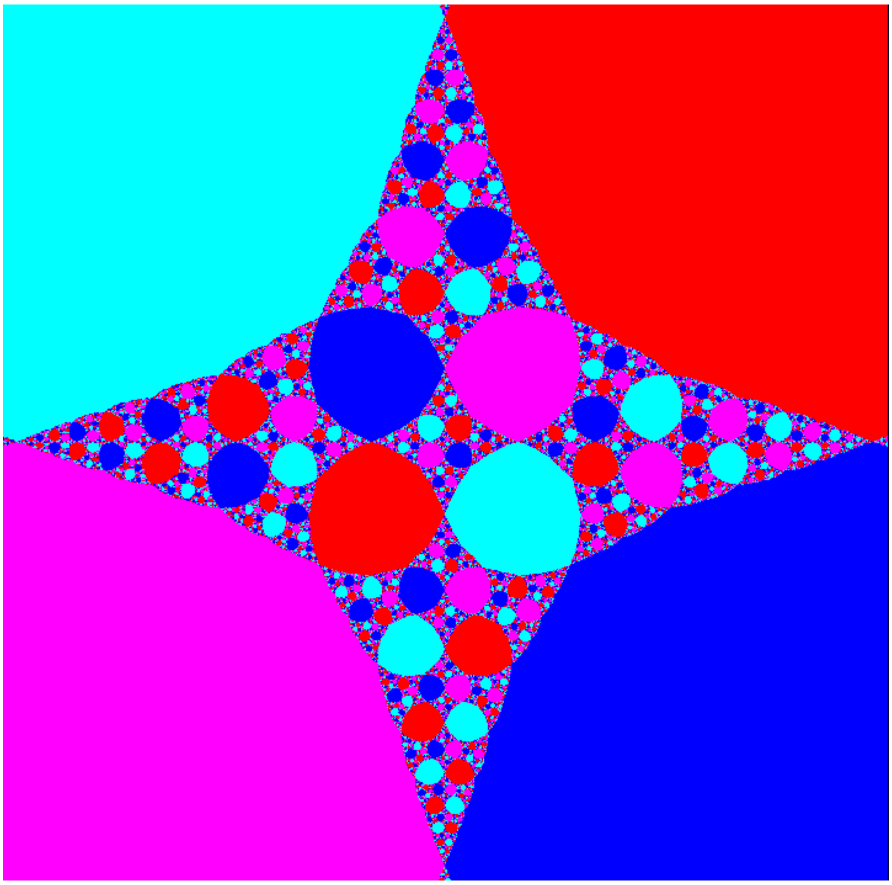

Fig. 6. Four basins of attraction for the octahedral 5-map

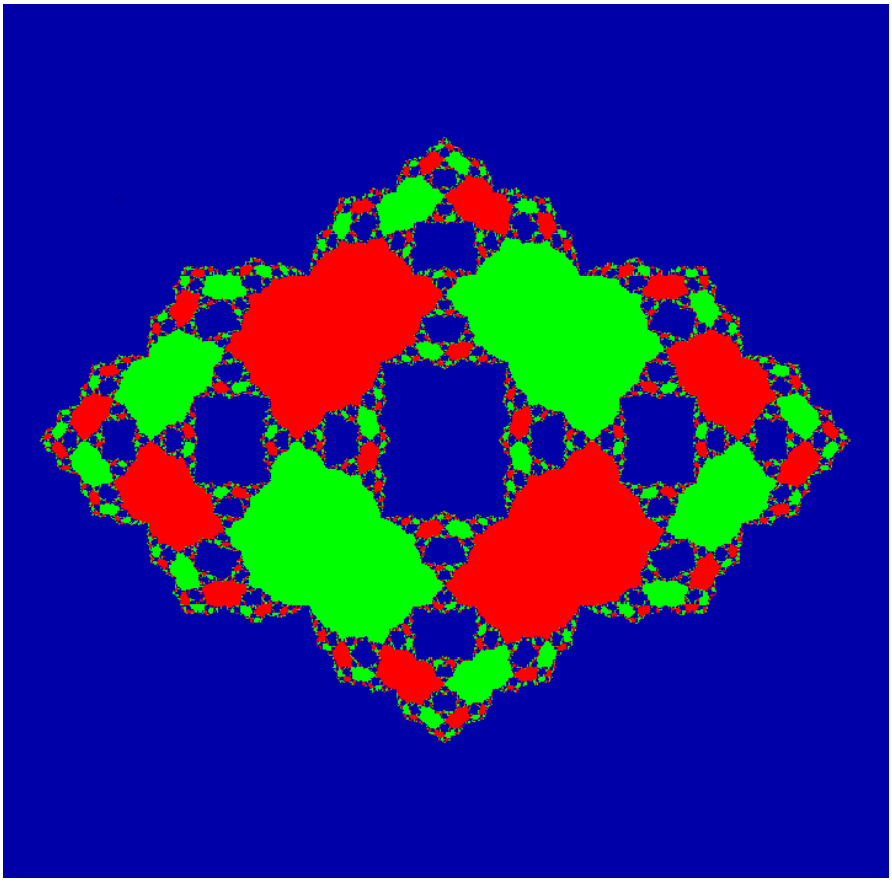

Fig. 7. Three basins of attraction for h_{11} restricted to a 15-line

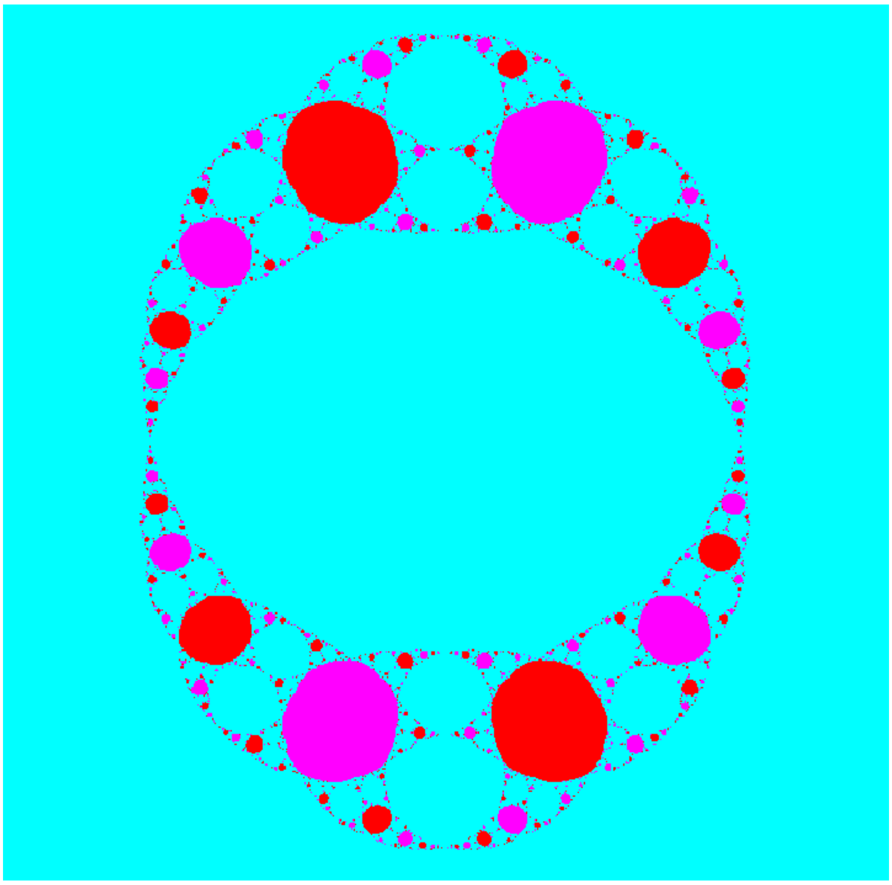

Fig. 8. Three basins of attraction for h_{11} restricted to a 30-line

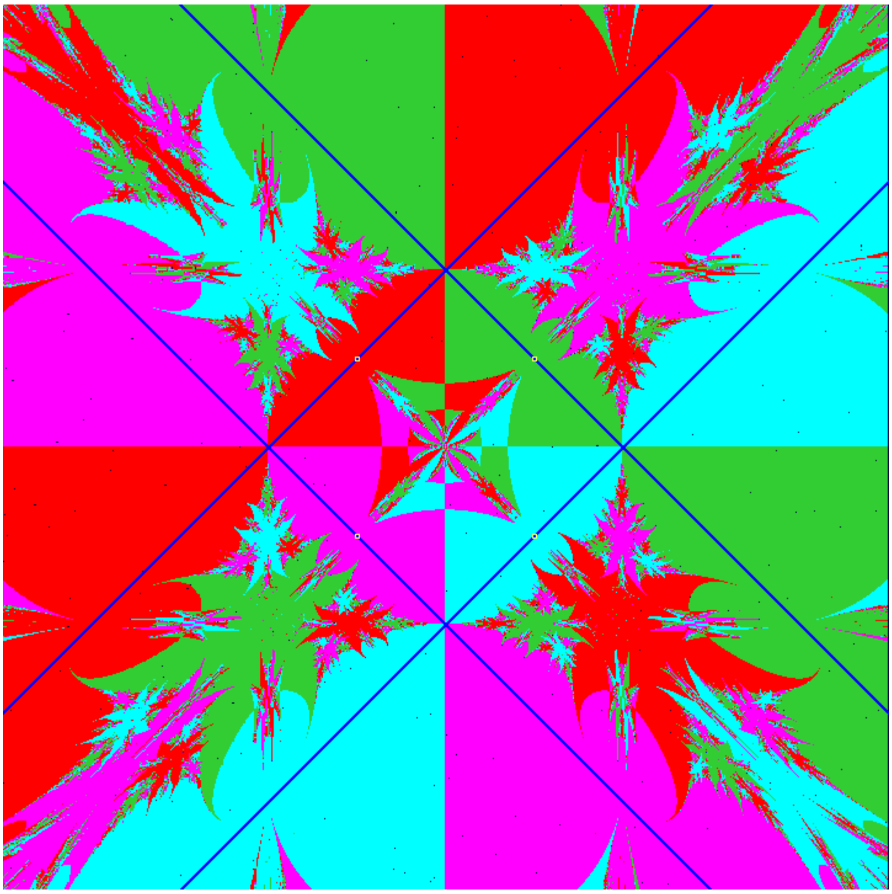

Fig. 9. Chaotic attractors for h_{11} on an $\mathbf{RP}^2$ with $\mathcal{S}_4$ symmetry

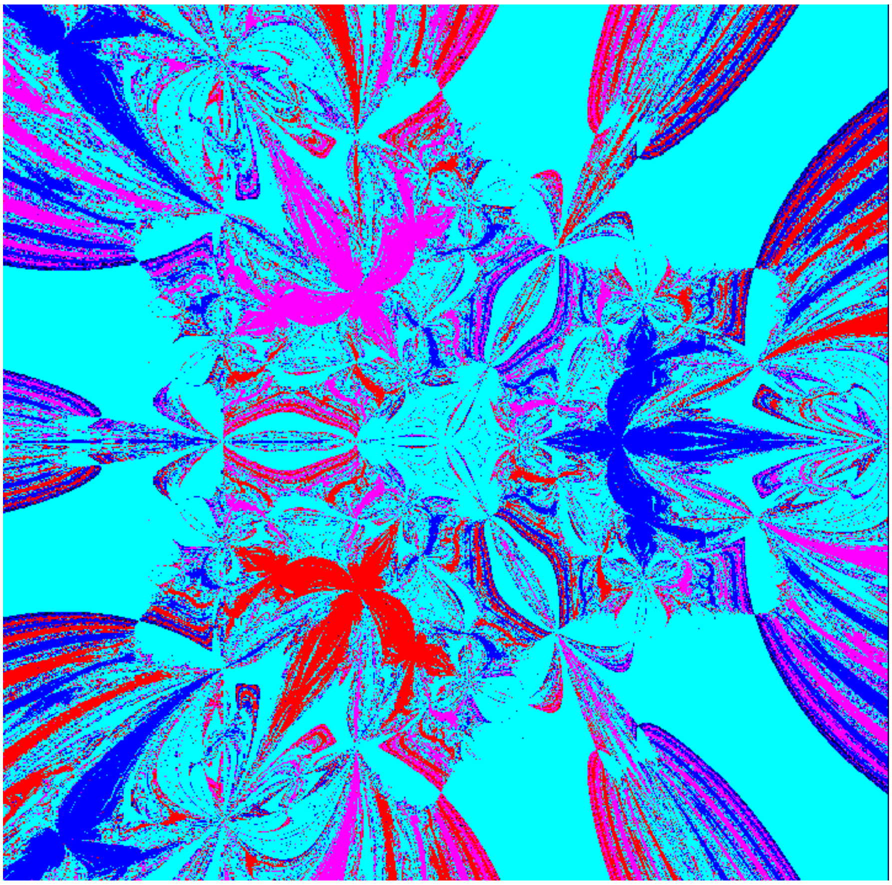

Fig. 10. Chaotic attractor for h_{11} on an $\mathbf{RP}^2$ with $\mathcal{S}_3$ symmetry

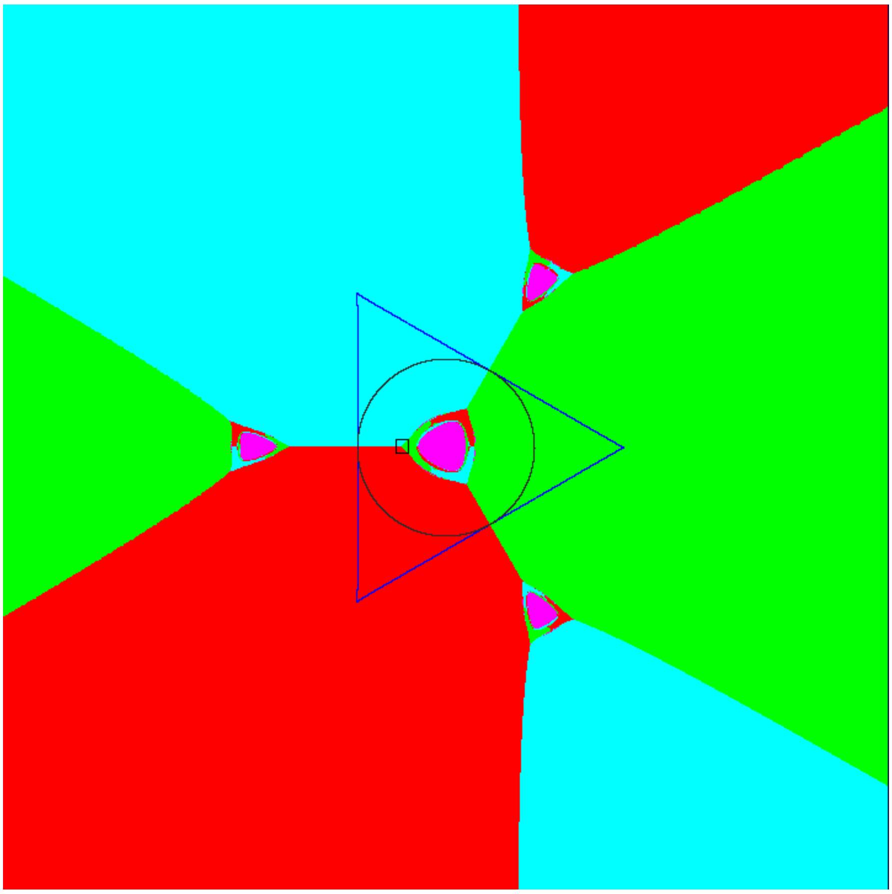

Fig. 11. Four basins of attraction for f_6 restricted to an $\mathbf{RP}^2$

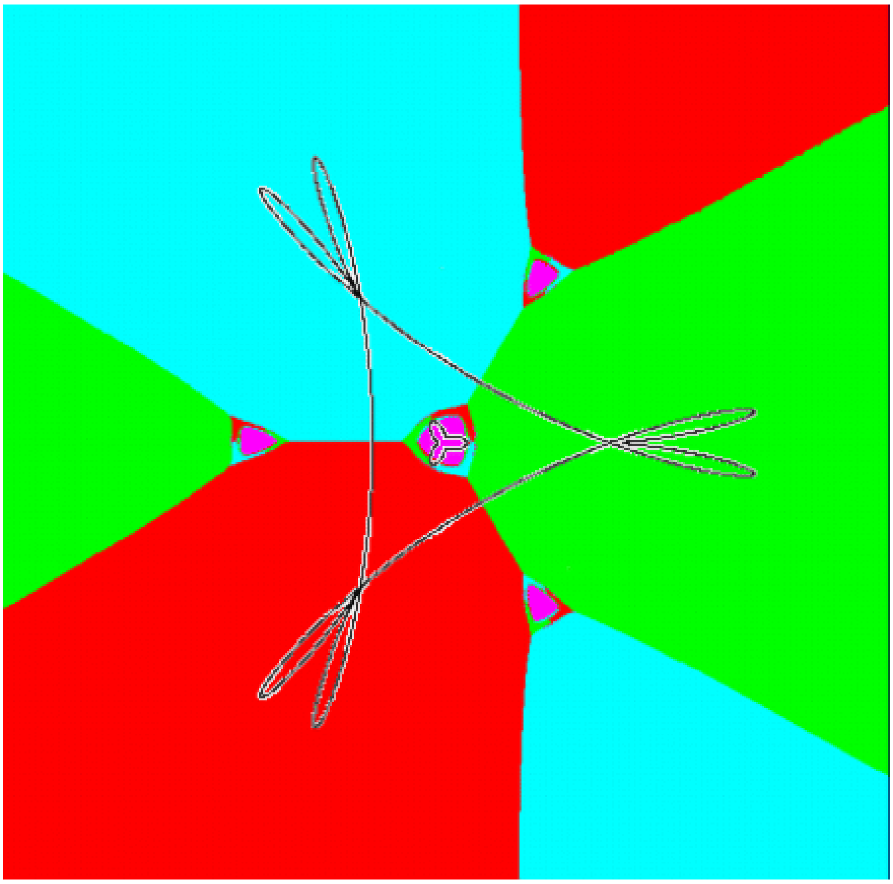

Fig. 12. Critical set of f_6 restricted to an $\mathbf{RP}^2$

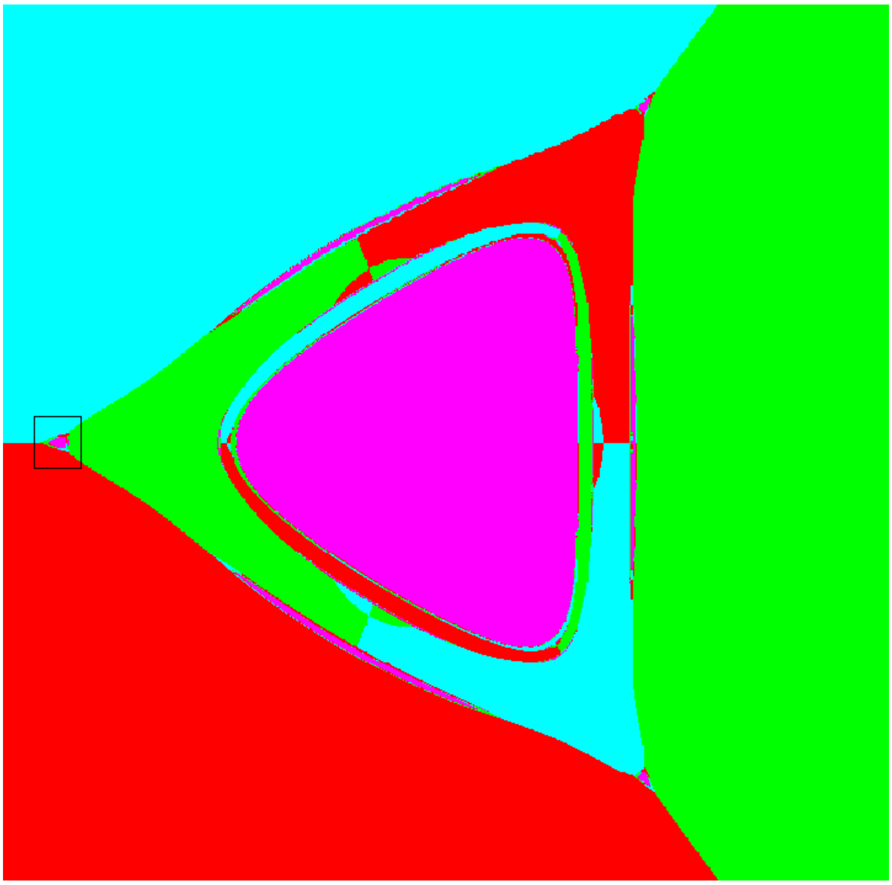

Fig. 13. Detail of the left cusp of central basins in Figure 11

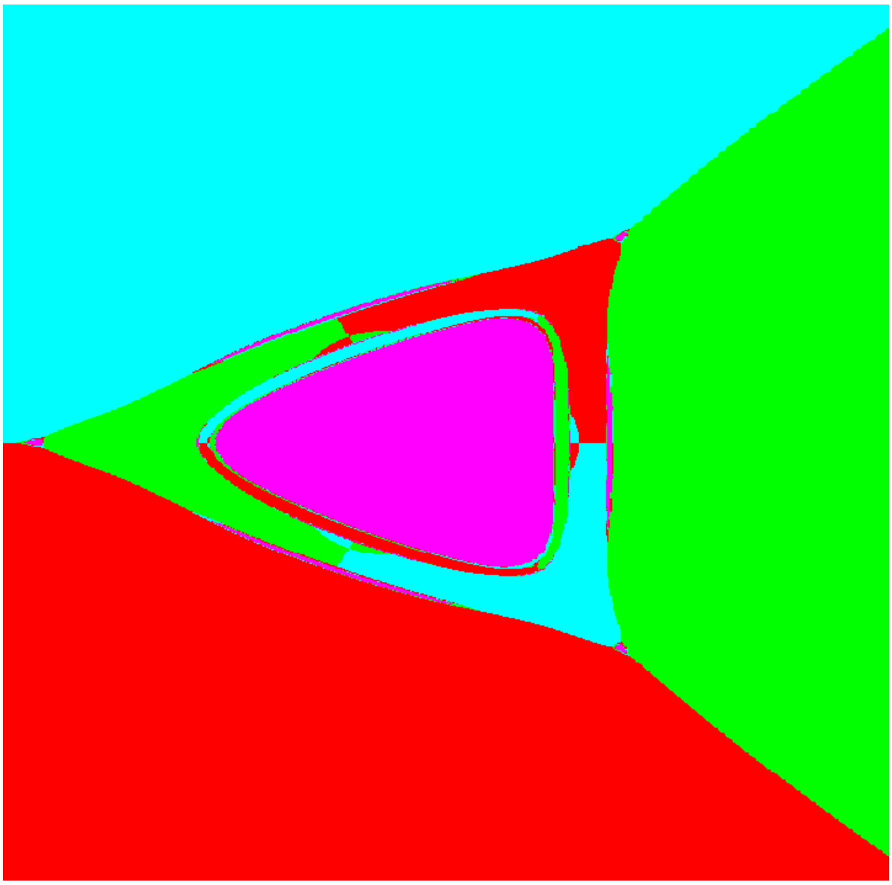

Fig. 14. Detail of the left cusp in Figure 13

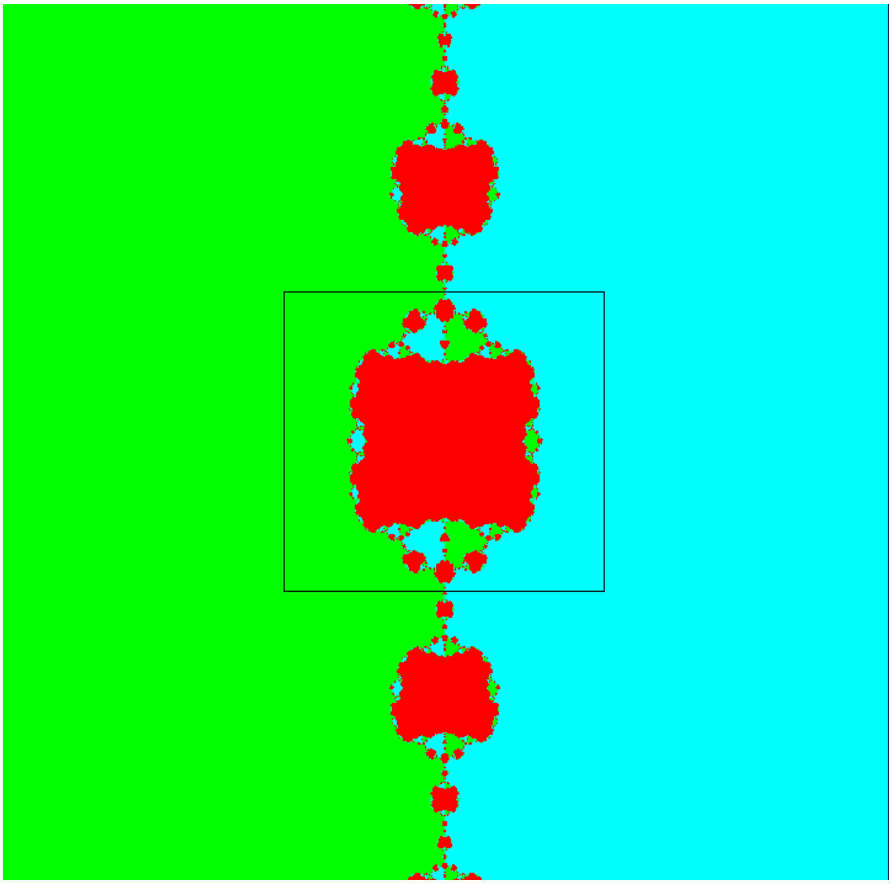

Fig. 15. Three basins of attraction for f_6 restricted to a 15-line

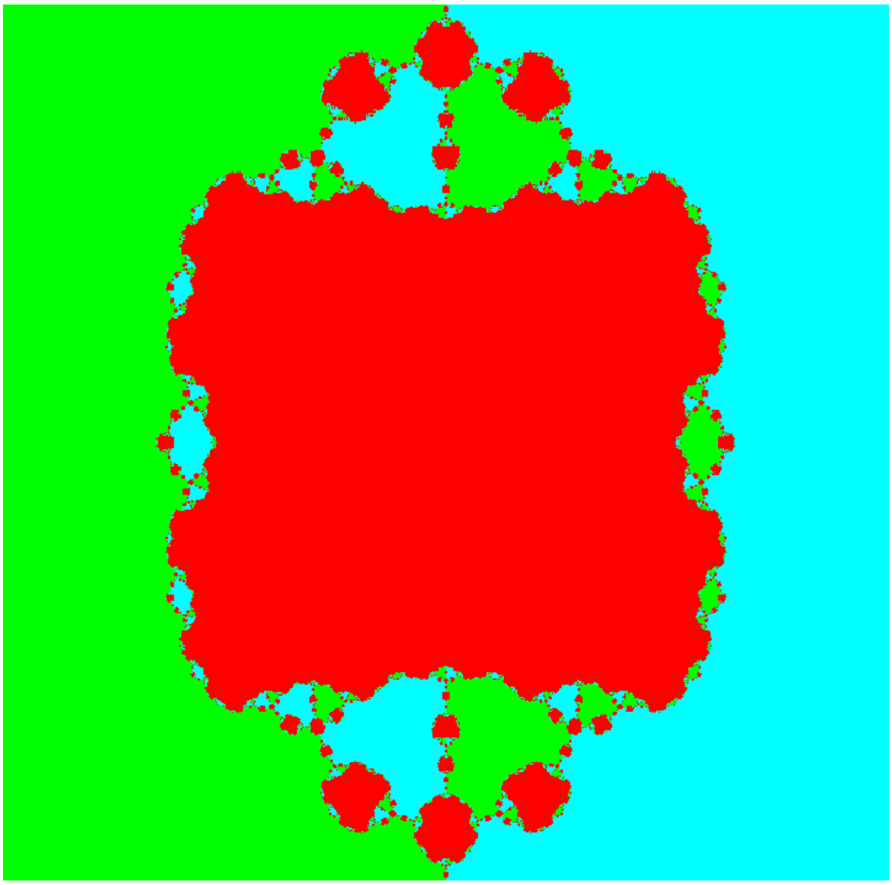

Fig. 16. Magnified view of the boxed region in Figure 15

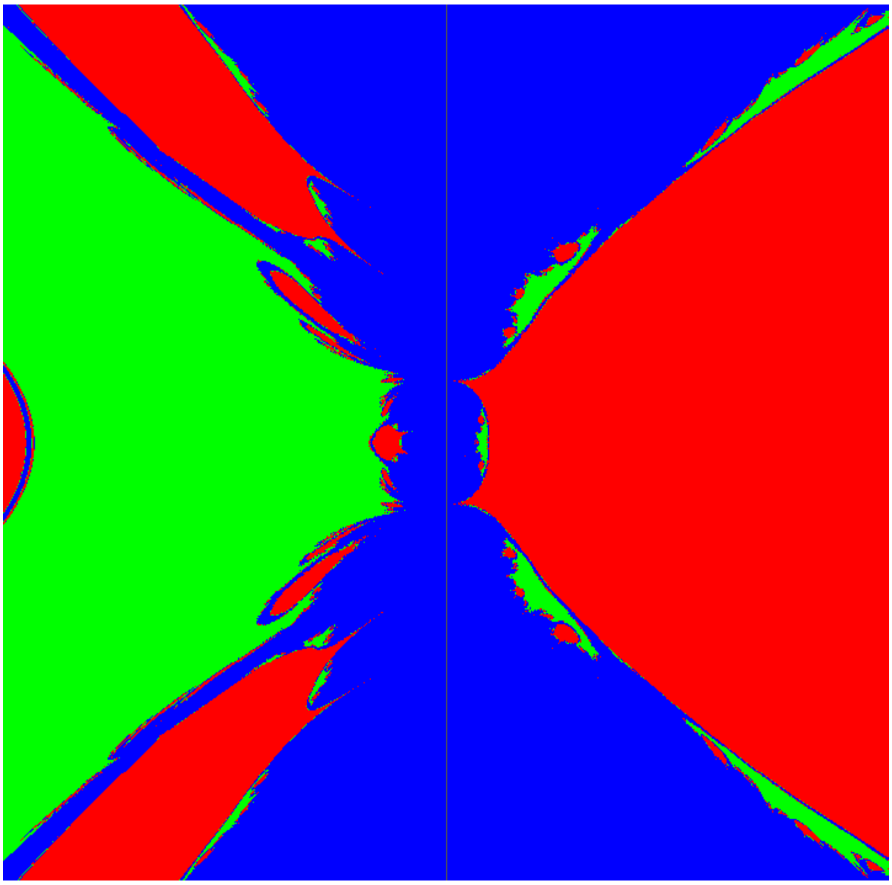

Fig. 17. Chaotic attractor for f_6 on an $\mathbf{RP}^2$

Maps with $\mathcal{A}_6$ Symmetry

Figure 18 When "restricted" to a 45-line, the degree-16 map "mostly" converges to one of the 45-points on the line. Does this occur for almost every point on the line? Do the black specks consist of points whose trajectories fail to converge to one of the four attracting 45-points that lie in the large "central" basins? The BAS algorithm checked 60 iterates before concluding that a trajectory did not converge.

Figure 19 The degree-19 map with icosahedral symmetry attracts almost all points in the sphere to an antipodal pair of vertices. Each of the six colors corresponds to such a pair and the three large basins each contain a vertex. For the conic-preserving h_{19}, the basin plot on each conic looks like this one. Moreover, each basin is the 1-dimensional intersection of a 2-dimensional basin in $\mathbf{CP}^2$.

Figure 20 The image shows the behavior of h_{19} on one of the 36 real projective planes determined by the basic conic-exchanging transformations. The large "radial" basins are immediate, that is, each contains one of the 72-points and come in pairs as do the period-2 attractors. Notice the repelling behavior along the 45-lines and particularly at their intersection in the 36-point $(0, 0)$.

Figure 21 Shown here are trajectories, colored according to their destinations, of the points in the vertical strip on the left. Many of the points in the strip map inside the "hazy pentagon" whose vertices lie on the 45-lines—the inner "star" is nearly filled. "Circumscribing" this pentagon is the outer star-like piece of the critical set shown in Figure 24. Furthermore, the pentagon seems to be the image of the inner pentagonal oval. Accordingly, the map folds the plane along the pentagon's edges just outside of which the 72-points make their presence seen in the dense streaks. Compare this pattern of streaks to that of the 72-lines shown in Figure 22. Figure 23 illustrates this local "squeezing" at a 72-point.

Figure 22 The lines tangent to a conic at the 72-points form an orbit of 72 lines. For any one of the 36 planes of reflection associated with transformations that exchange the systems of conics—call such a plane $\mathcal{R}$, there are five of the 72-lines that intersect $\mathcal{R}$ in a real projective line. The picture shows their configuration in the plane of Figures 20 and 21. Each pair receives a single color according to the scheme of the basin plots. A given pair passes through the associated pair of 72-points; they intersect in the corresponding repelling and fixed 36-point.

Figure 23 The green horizontal line corresponds to the $\mathbf{RP}^1$ intersection of the reflection plane $\mathcal{R}$ and the 36-line passing through the pair of green 72-points from the basin plot. The dark curve is where h_{19} sends the line. Sitting at the sharp cusps are the 72-points which the map exchanges. As indicated in the caption to Figure 21, the line folds over at these critical points. The upper two sharp turns are not critical values; they occur where the line passes through the yellow and red "streak" that approximate 72-lines.

Figure 24 Here is a *Mathematica* contour plot on $\mathcal{R}$ of the sixth degree curve—the set of points in $\mathcal{R}$ that satisfy the equation $F_6 = 0$. The critical set of h_{19} in $\mathbf{CP}^2$ meets $\mathcal{R}$ in this curve. The superattracting 72-points are the inflection points.

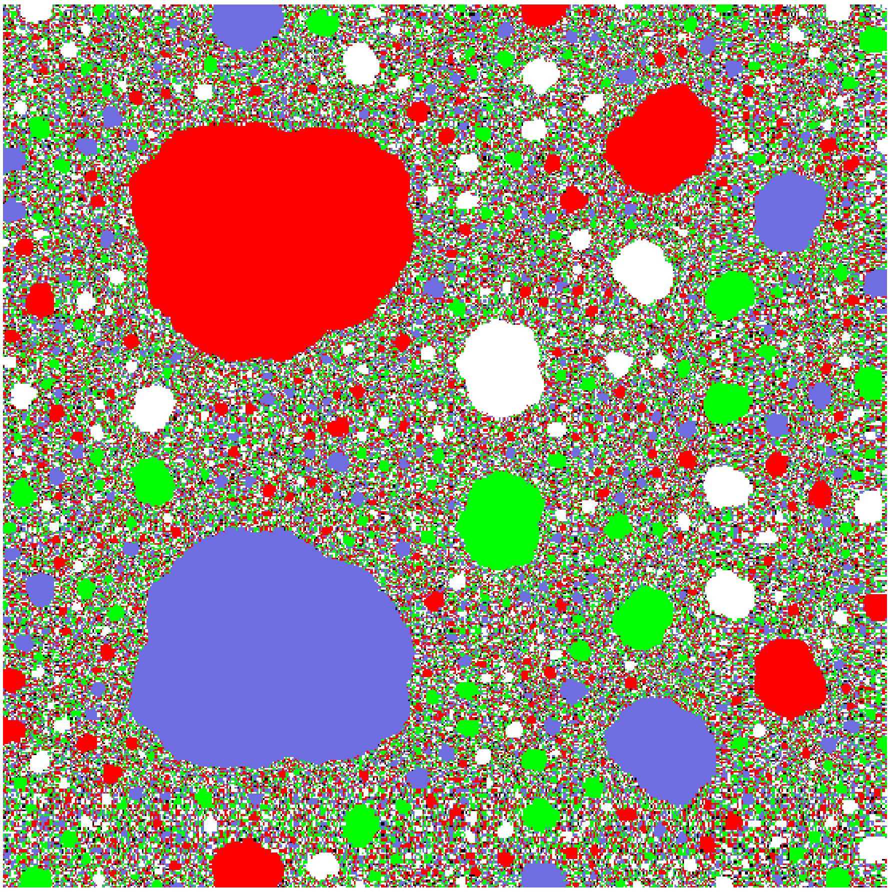

Fig. 18. Dynamics of the 16-map

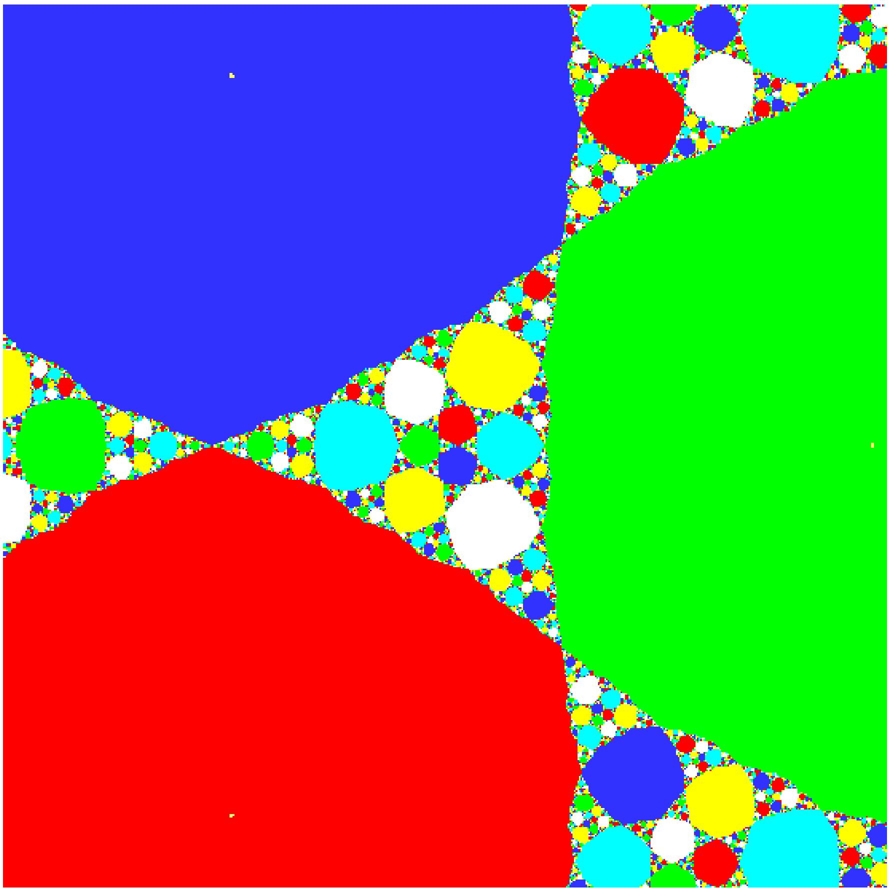

Fig. 19. Icosahedral dynamics of the 19-map

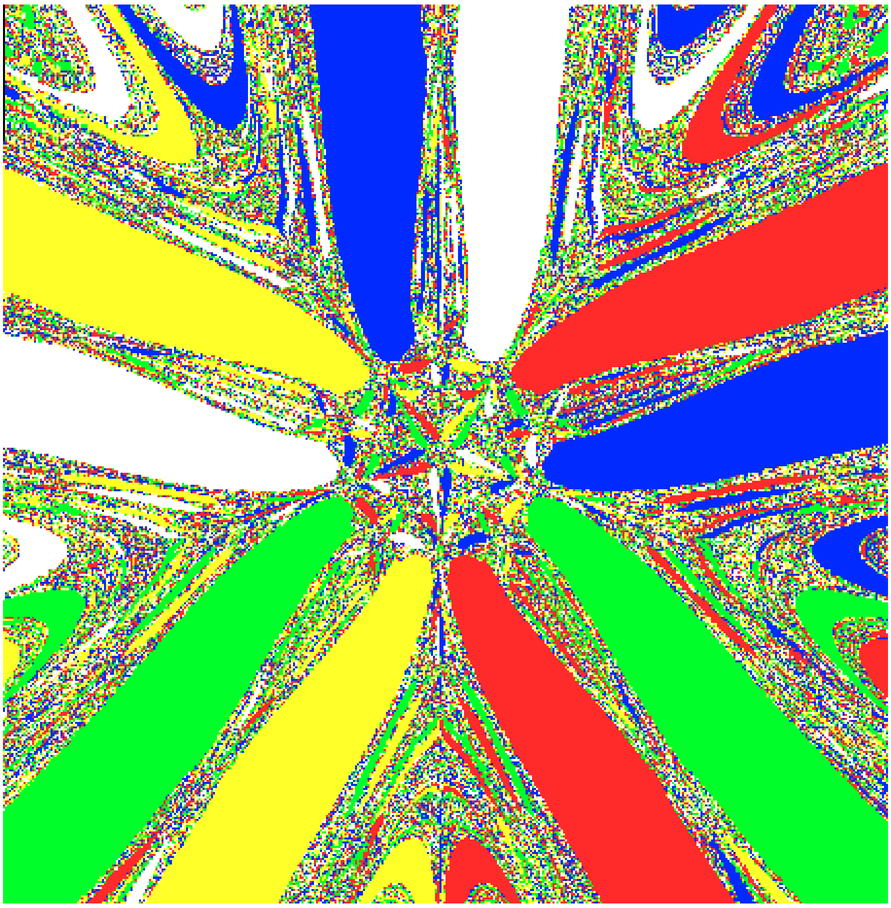

Fig. 20. Dynamics of h_{19} on a special real projective plane

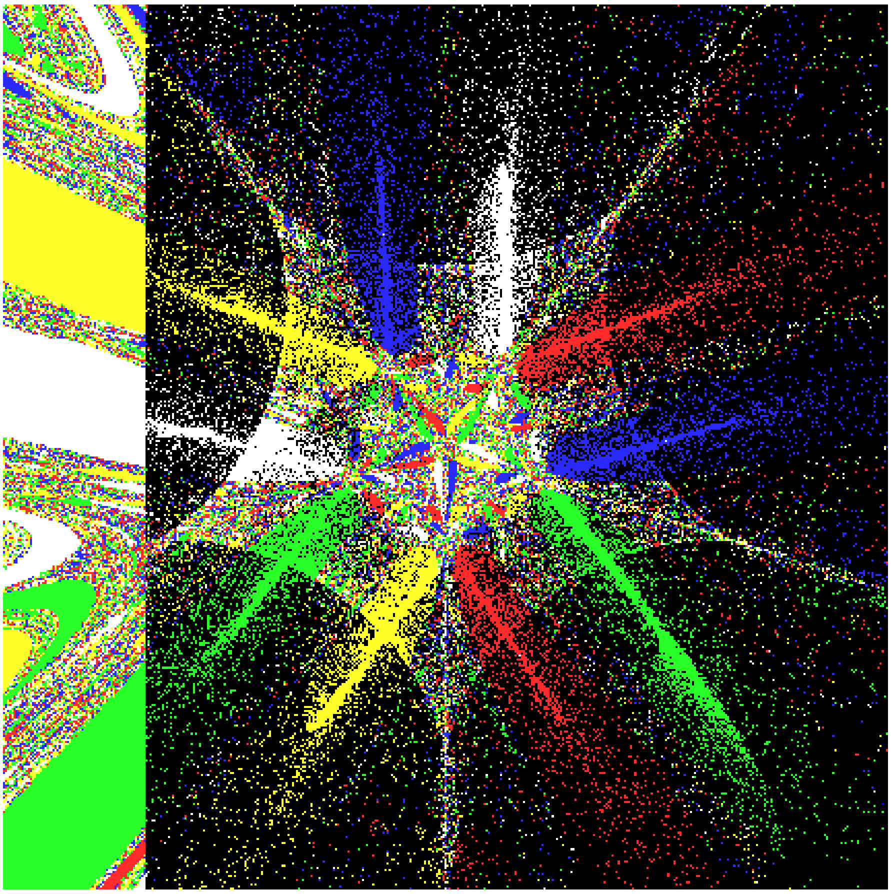

Fig. 21. Dynamics of h_{19} on a special real projective plane

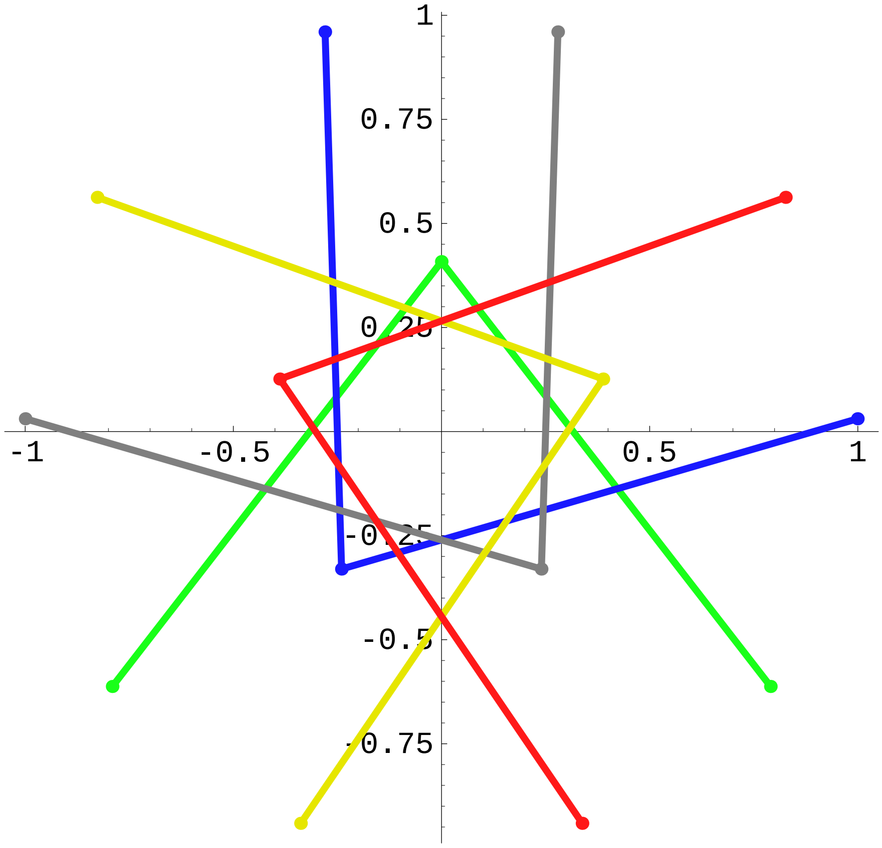

Fig. 22. Configuration of 72-lines

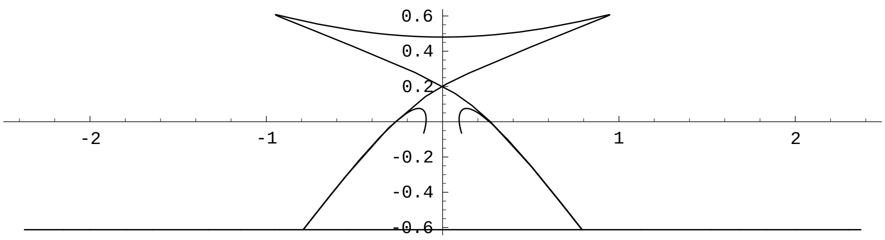

Fig. 23. Image of a 36-line under h_{19}

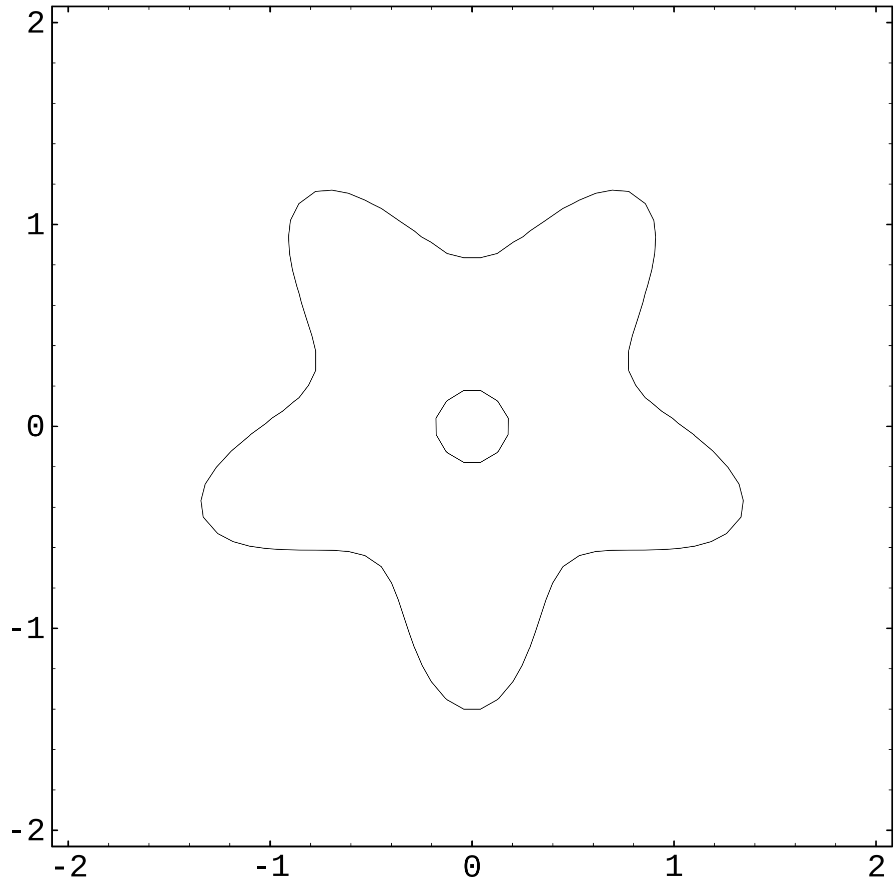

Fig. 24. Critical set of h_{19}

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