

Experimental Mathematics

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Abstract Computer tools (ray-tracing software, 3D modeler), new technologies (laser cutting, 3D printing), new communities (fablabs), increasing computing power and improved graphic screens: all together make it possible for the mathematicians to give life to their ideas. Moreover, having described basic mathematical objects, one can play with their parameters and just see what arises. This paper illustrates this fact through several examples: anamorphoses, cardioid based structures and circle packings.

1 Introduction

We can assume, with strong confidence, that Gaston Julia (1893-1978) never saw the image of a Julia set. Felix Klein (1849-1925) and Robert Fricke (1861-1930) drew only one representation of a limit set[8]. Some thirty years ago, on the 1st of March 1980, Beno t Mandelbrot first saw a blurry image of the fractal set christened after him appearing on his printer. He then spent several weeks making new images, to understand the shape he just discovered[14]. Nowadays, anyone can write or download a program or a smartphone application, running orders of magnitude faster, zooming to scales out of reach at Mandelbrot's time. Playing with mathematical objects is easier today, scientists can experiment more sophisticated structures in reasonable time. Any mathematician with elementary programming skills can imagine a new shape, a new curve, translate it into a program and see it appearing within seconds on his high resolution screen, using a standard computer. Exploring the field of mathematical objects becomes easy and affordable due to several factors:

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- The increasing power of processors, which makes calculations easier to perform.
- High resolution of computer screens, which gives precise and realistic images.
- High level programming languages which let the programmer focus on the heart of the problem to solve.
- Specialized computer applications like ray-tracing software or 3D modelers, when mastered, can also alleviate the mathematician’s work. Those softwares are available in open-source and free versions.
- Laser-cutting machines, 3D printers are valuable tools to help making real-world versions of mathematical shapes.

In the rest of the paper, I will present several categories of works achieved by using part or all of those tools, and I try to show how, from the initial object I wanted to visualize, I was able to tune the parameters in order to obtain a whole family of related objects.

2 Anamorphoses

2.1 Definition

Anamorphoses are distorted images (resp. objects) needing to be seen through a mirror, and from a specific point of view, in order to reconstruct the original image (resp. objects). Anamorphoses appeared in Europe at the time of Renaissance, when artists and scientists discovered the laws of perspective, and independently in China during the XVIth century. A complete history of anamorphosis can be found in Jurgis Baltrušaitis’ *Anamorphoses ou Thaumaturgus opticus* [3].

There are different methods for producing anamorphoses, either analytical, empirical or hybrid.

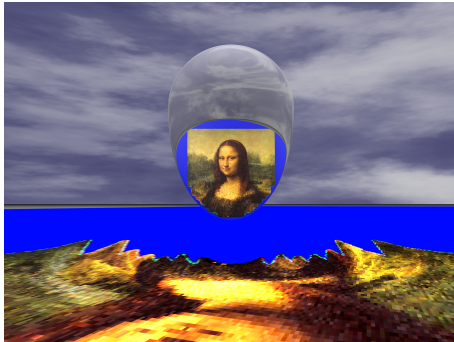


Fig. 1 Anamorphosis in an egg-shaped mirror.

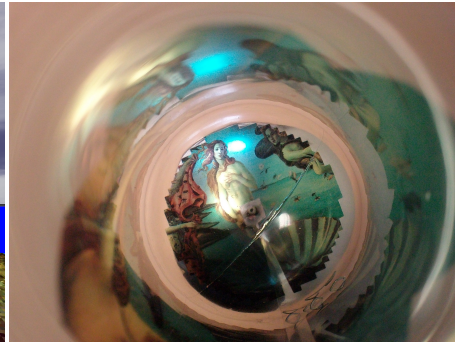


Fig. 2 Distorted image on a cylinder



Fig. 3 3D anamorphic sculpture.

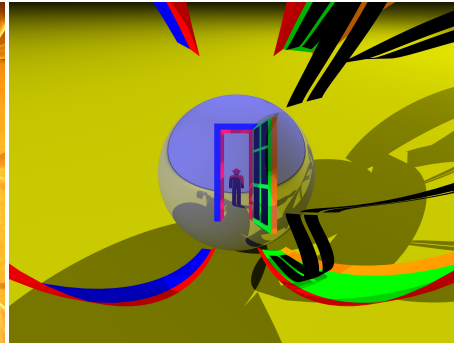


Fig. 4 Distorted quadrilaterals (preliminary try).

2.2 A Computer Method and its Evolution

In [5], I described a procedure for testing and constructing catoptric anamorphoses in the general case. The setting-up of an anamorphosis needs three items : a mirror, an observer, and the locus where the distorted image lays, which I will name *surface of distortion*. This method can compute the distorted image corresponding to an anamorphosis setting and print it, in order to obtain a real size version of the anamorphosis. In [5], the surface of distortion was either a flat or a developable surface. Extensions of the method can also help define anamorphoses where the surface of distortion is no longer developable: we then lose the possibility of printing the distorted design, but are still able to achieve it by directly drawing on the surface, using the information returned by the program. In October 2010, James Hopkins, a British sculptor [9], asked me whether it would be possible to compute the shape of a three-dimensional wired form sculpture which would represent a chair when seen through a spherical mirror. This led to an improvement of the original method[6], where the surface of distortion is replaced with a set of volumes of distortion. The new method computes the distorted image of a line (either a line segment or a circle arc), and gathers all those images in a single three-dimensionnal anamorphic sculpture. Then exporting the results of those computation to a file, we can build a real-world three-dimensionnal object, using 3D printing techniques. Figure 3 shows an example of this process. Yet another improvement of the method is under development, in collaboration with James Hopkins, allowing the definition of 3D distortion of plain quadrilaterals (see figure 4). The related real-world sculpture is under construction (november 2013).

In conclusion, when a method, and the tools used in it are well mastered, improvements and evolutions are easy to implement. One can imagine possible improvements, then test them first virtually and validate them with real-world achievements.

3 Cardioid variations

The cardioid is a very old and known curve, which can be defined in several manners. Pedoe describes a method for constructing a cardioid as the envelop of a set of circles:

- Draw a circle and choose a point on its circumference.
- Draw circles with centers lying on the initial circle, and passing through the chosen point.
- The envelop of this set of circles is a cardioid[12] (see figure 5).

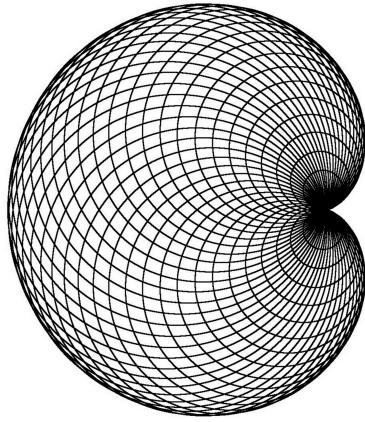


Fig. 5 Pedoe method for cardioid.

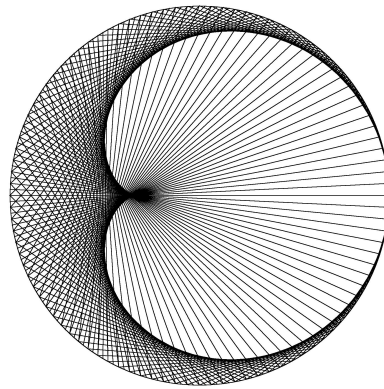


Fig. 6 String method for cardioid.

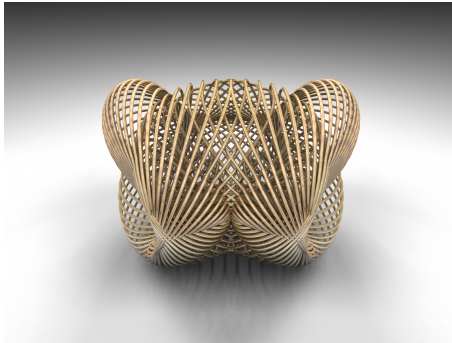


Fig. 7 variation from Pedoe method.

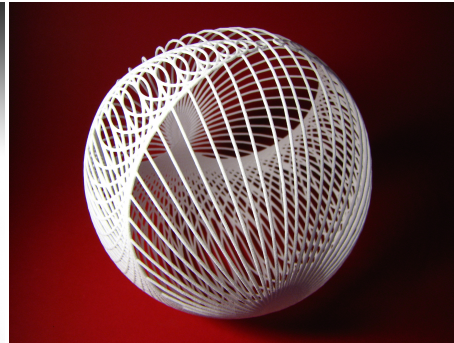


Fig. 8 Variation from string method.

But the result is flat. What if we rotate each circle in the third dimension, with an angle depending on its radius ? The function relating the rotation angle to the radius can be arbitrary, and each choice defines a different final shape (see figure 7). The shape is not difficult to code, once drawn. We can output the information (circle centers, radii, angles) and use that to obtain a real three-dimensional version of the virtual drawing. The experience of turning this 3D cardioid in your hands is still stronger than seeing it on the screen. Each angle of vision makes it look different, and the observer finds new symmetries each time he moves it.

Playing with the function defining the rotation of each elementary circle, one can obtain an infinity of different shapes, all very different from each other. The best way to investigate this family of shapes is either intensive tests, or animations. Intuition alone may miss interesting structures.

Another way to draw a cardioid is the following algorithm, whose result is shown on figure 6 :

- Draw a circle.
- All around the circumference of this circle, draw n equidistant points.
- Numbering those points $1 \dots n$, draw line between points i and $(2 \times i) \bmod(n)$

Once again, the result is flat: we can now try to replace each line segment with a torus whose diameter equals the length of the segment, and that produces the object of figure 8. Surprisingly, the external shape of this object is a simple sphere. Simple mathematical reasoning might prove this, but the fact that it was *discovered* while drawing it is a strong argument in favour of experimental mathematics. A lot of explorations are still to be done: for example, what if the couple of points defining a line segment is changed from $(i, 2 \times i)$ to $(k_1 \times i, k_2 \times i)$? There is so much to explore that even those easy programming tasks are not yet achieved.

4 Circle packings

Circle packing can be seen as the art of placing tangent circles on the plane, leaving as little unoccupied space as possible. Circle packing has been (re)introduced by William Thurston [15] in 1985. Kenneth Stephenson developed its study in [13]. In this section, I will show how one can use different ways of producing circle packings, together with different geometric transformations that preserve the tangency property of the arrangement, in order to produce elegant and appealing images. For sake of aesthetical homogeneity, I will only consider packings of tangent circles included in one external circle, also part of the tangency pattern. I mainly use two operations : defining Steiner chains, and using "Apollonian Gasketization" to fill the gaps between the generated circles. At Bridges 2012, Inglis and Kaplan [10] presented a method to produce fractal circular rings of tangent circles, where spaces are filled with Apollonian circles. Their algorithm shares some similarity with ours, but is in some ways more restrictive.

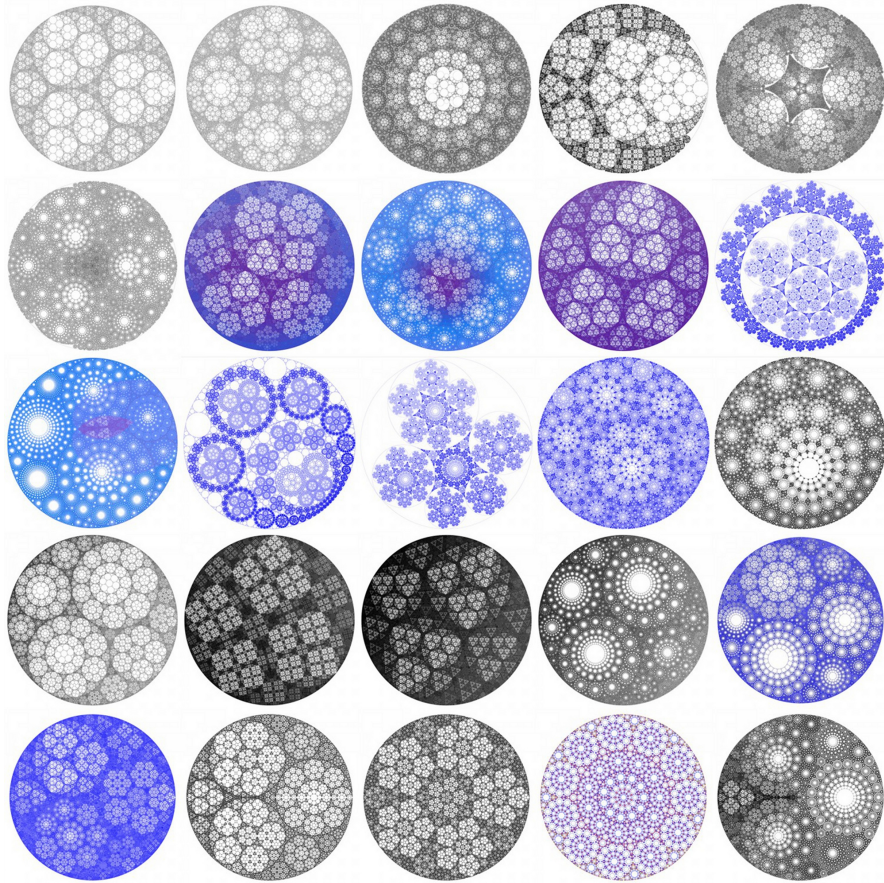


Fig. 9 Circle packing experiments.

4.1 Steiner Chains

Steiner chains are chains of tangent circles, each of those circles also tangent to two fixed and non intersecting circles. Steiner chains are obtained by first constructing the easy solution: n circles forming a chain between two concentric circles, then using a circle inversion to distort the arrangement (since circle inversion preserves tangency). The way one chooses the inverting circle is a parameter of the design, which can be used to create an infinity of different shapes. The number n is also a parameter that leads to different solutions.

4.2 Apollonian Gasket

Apollonian gaskets are obtained by recursively filling the gaps between three tangent circles with a circle tangent to all those three circles. The standard Apollonian Gasket starts with three equal circles inside a fourth one. In our procedure, we consider filling the gaps between any set of three tangent circles.

4.3 Putting Things Together

The procedure used to generate circle packing patterns can be summarized in one sentence: "*Each time you generate a new circle, fill it with a Steiner chain, then fill the gaps by mean of gasketization*". Each *steinerization* might have its own parameters. Such a simple construction rule produces a very rich set of different patterns. We can still add more diversity by applying at the end of the procedure a geometric transform that preserves the tangency property, like Möbius transform¹ or circle inversion. A complete exploration of the graphical potentialities of Möbius transforms can be found in Mumford's Indra's Pearls [11]. Figure 9 shows some variants. More are available in [7], and still more of them are to be discovered.

5 Conclusion

Using those three examples, I wanted to show that wide fields of mathematical art can be cleared by random exploration. The programmer can launch his program, looking at the image appearing on his screen. He can first compute a low resolution image, or abort the drawing if he is not satisfied with what he sees, refine the parameters of the object, and launch again. All this in seconds. One can even imagine a cooperative and parallel exploration: by making available a smartphone application where the user can launch a design, with his own parameters, collect the results, ask people to rate them. This could reveal hidden design rules and interesting properties. Several authors developed dedicated softwares to explore mathematical objects. One can cite Ken Stephenson[2] in the field of circle packings, and Phillip Kent[1] for conical, cylindrical and pyramidal anamorphoses.

¹ Möbius transforms were in fact first described by Euler in *Acta Academiae Scientiarum Petropolitanae*, in 1777 (see [4])

References

1. Anamorph me! website (with software).
<http://www.anamorphosis.com>
 (accessed 18/11/2013).
2. Ken Stephenson's website (with circle packing software).
<http://www.math.utk.edu/~kens/CirclePack/>
 (accessed 18/11/2013).
3. Jurgis Baltrušaitis. *Anamorphoses ou Thaumaturgus Opticus*. Id es et Recherches. Flammarion, 1984.
4. Claude Bruter. Complex history, Letters to the Editor. *Notices Amer. Math. Soc.*, 54(8):958, 2007.
5. Francesco De Comit . A General Procedure for the Construction of Mirror Anamorphoses. In George W. Hart and Reza Sarhangi, editors, *Proceedings of Bridges 2010: Mathematics, Music, Art, Architecture, Culture*, pages 231–238, Phoenix, Arizona, 2010. Tessellations Publishing.
6. Francesco De Comit . A New Kind of Three-Dimensional Anamorphosis. In Reza Sarhangi and Carlo H. S quin, editors, *Proceedings of Bridges 2011: Mathematics, Music, Art, Architecture, Culture*, pages 33–38, Phoenix, Arizona, 2011. Tessellations Publishing.
7. Francesco De Comit . Circle Packing Explorations. In George W. Hart and Reza Sarhangi, editors, *Proceedings of Bridges 2013: Mathematics, Music, Art, Architecture, Culture*, pages 399–402, Phoenix, Arizona, 2013. Tessellations Publishing.
8. R. Fricke and F. Klein. *Vorlesungen  ber die Theorie der Automorphen Functionen*. Number vol. 1 in *Vorlesungen  ber die theorie der automorphen functionen*. B.G. Teubner, 1897.
9. James Hopkins. James Hopkins' website. <http://www.jameshopkinsworks.com>
 (accessed 09/09/2013).
10. Tiffany C. Inglis and Craig S. Kaplan. Circle patterns in gothic architecture. In Robert Bosch, Douglas McKenna, and Reza Sarhangi, editors, *Proceedings of Bridges 2012: Mathematics, Music, Art, Architecture, Culture*, pages 133–140, Phoenix, Arizona, USA, 2012. Tessellations Publishing.
11. David Mumford, Caroline Series, and David Wright. *Indra's Pearls: the Vision of Felix Klein*. Cambridge Univ. Press, Cambridge, 2002.
12. Dan Pedoe. *Circles: A Mathematical View (Spectrum)*. The Mathematical Association of America, 2 edition, 1997.
13. Kenneth Stephenson. *Introduction to Circle Packing. The Theory of Discrete Analytic Functions*. Cambridge Univ. Press, Cambridge, 2005.
14. I. Stewart, A.C. Clarke, and N. Lesmoir-Gordon. *The Colours of Infinity: The Beauty and Power of Fractals*. Clear Books, 2004.
15. William Thurston. The Finite Riemann Mapping Theorem, 1985. Invited talk, Purdue University.