POLYHEDRAL EVERSIONS OF THE SPHERE. FIRST HANDMADE MODELS AND JavaView APPLETS

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Abstract

This article gives the tools for self-construction of the polyhedral models which appear during the process of everting the polyhedral sphere. It can be understood as a pedagogical device to understand the different steps of that process.

1 Introduction

There are problems which are real challenges. The sphere eversion problem belongs to that category! How is it possible to exchange the internal and the external face of a sphere without tearing or folding its surface? At first sight, it seems impossible! But, by authorizing the surface to cross itself and by respecting rules established by mathematicians, it becomes possible and the eversion[1] can be shown on the screen of a computer. At the conference, I presented pictures of polyhedra imagined by the blind mathematician Bernard Morin which illustrate the sphere eversion. The approach which is developed here allowed the discovery of the first *eversion of the cuboctahedron*. We present three introductory models which lead directly to the central stage of the eversion.

The starting point of this collection of models is a minimal Boy surface with 9 vertices inspired by Ulrich Brehm's work [1]. It is a non-orientable surface which presents a threefold axis of symmetry. Some of its faces intersect themselves giving birth to an intersection line and a *triple point*. The same construction process can be applied to get a model with a fourfold symmetry called *open halfway-model*. Then the surface becomes orientable and has a *quadruple point*. The reader is invited to build by himself these two first models. The third model, called *closed halfway-model*, reaches the necessary level of complexity to carry out the eversion of the cuboctahedron. Handmade models, photos in artificial light and JavaView[2] applets were used to highlight the thought of the blind mathematician.

[1] http://torus.math.uiuc.edu/optiverse/

[2] http://www.javaview.de/

For the exchanges with Morin, we used a closed halfway-model in white drawing paper, less aesthetic than the one presented here. Impossible to see inside! The construction had to be improved with transparent faces in rhodoïd and with the use of two different colours (red and blue) for each side of the opaque faces. Many years later, the models were realized on computer with Konrad Polthier's software JavaView; all the details of the eversion can be shown. The two sides of a face can be displayed with two different colours as on the handmade models. Furthermore, JavaView is able to handle the intersections; the triple point and the quadruple point are immediately visualized as intersections of three or four faces.

2 Minimal polyhedral Boy surface

At the beginning of the 1980's, during summer holidays readings I fell casually on blind mathematician Bernard Morin's article "Le retournement de la sphère" [2] illustrated by Jean-Pierre Petit's drawings in the revue Science. It aroused my curiosity and I tried to understand step-by-step the sphere eversion[3] that Morin had imagined. A few years later (1986), I met him at the University Louis Pasteur of Strasbourg. At the end of the formation he invited me, with other colleagues, in his office to show us a great wire model of the Boy surface ([3], [4]). I immediately recognized – and was fascinated by – the surface I discovered a few years earlier in his article. After our discussion, he gave me a letter written by Ulrich Brehm which contained a short description of a Boy surface with nine vertices. It was a variant of the minimal Boy surface conceived by Brehm [1] which Morin has adapted to the polyhedral sphere eversion. I tried to build it briefly and succeeded after a few days.

2.1 Construction of a polyhedral Boy surface

Boy surfaces[4] are obtained by gluing together a Möbius band[5] and a disk along their boundaries. The first model we will describe is Ulrich Brehm's polyhedron[6]. Its Möbius band is a three half-turns twisted band; it is a remarkable assembly of three concave pentagons which is explained below.

The first pentagon $P_0 = C_0A_0B_0A_1B_1$ respects the following conditions :

1. the triangle $A_0B_1C_0$ is equilateral,
2. the point B_0 is its orthocenter,
3. the vertex A_1 is so that the quadrangle $C_0B_0A_1B_1$ is a parallelogram.

[3]http://www.lutecium.org/jp-petit/science/maths_f/Retournement_sphere/PLS_79.pdf

[4]http://arpam.free.fr/The%20Boy%20Surface%20as%20Architecture%20and%20Sculpture.pdf

[5]http://www.mathcurve.com/surfaces/mobius/mobius.shtml

[6]http://www.mathcurve.com/polyedres/brehm/brehm.shtml

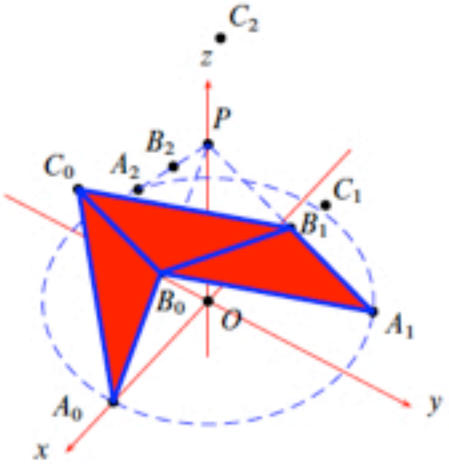

Fig. 1 The pentagonal face P_0 and its triangulation. Note that the three triangles in red are isosceles and have an angle which measure is 120°. Simple and nice! During the deformation the pentagons can be folded along the sides of the triangles. This picture is realized with LaTeX and the packages pstricks and pst-3dplot.

Similarly, we construct two other pentagons $P_1 = C_1A_1B_1A_2B_2$ and $P_2 = C_2A_2B_2A_0B_0$. These 3 pentagons lean towards the faces of a regular tetrahedron $PA_0A_1A_2$.

The coordinates of the nine vertices are:

$$
\begin{array}{lll}
A_0(1;0;0) & A_1(-\frac{1}{2};\frac{\sqrt{3}}{2};0) & A_2(-\frac{1}{2};-\frac{\sqrt{3}}{2};0) \\
B_0(\frac{1}{2};0;\frac{\sqrt{2}}{2}) & B_1(-\frac{1}{4};\frac{\sqrt{3}}{4};\frac{\sqrt{2}}{2}) & B_2(-\frac{1}{4};-\frac{\sqrt{3}}{4};\frac{\sqrt{2}}{2}) \\
C_0(\frac{3}{4};-\frac{\sqrt{3}}{4};\sqrt{2}) & C_1(0;\frac{\sqrt{3}}{2};\sqrt{2}) & C_2(-\frac{3}{4};-\frac{\sqrt{3}}{4};\sqrt{2})
\end{array}
$$

If we consider the assembly $P_0 \cup P_1 \cup P_2$ we get a polyhedral Möbius band. We just have to add 7 triangles which assembly is homeomorphic to a disk (see Fig. 2-b):

- three dorsal triangular faces $Q_0 = C_0B_1A_2$, $Q_1 = C_1B_2A_0$ and $Q_2 = C_2B_0A_1$; their intersection is the triple point,
- three ventral triangular faces $R_0 = C_0A_2A_0$, $R_1 = C_1A_0A_1$ and $R_2 = C_2A_1A_2$,
- and to finish the equilateral triangle $A_0A_1A_2$.

Fig. 2 Construction of the Boy surface by gluing together a Möbius strip and a disk.

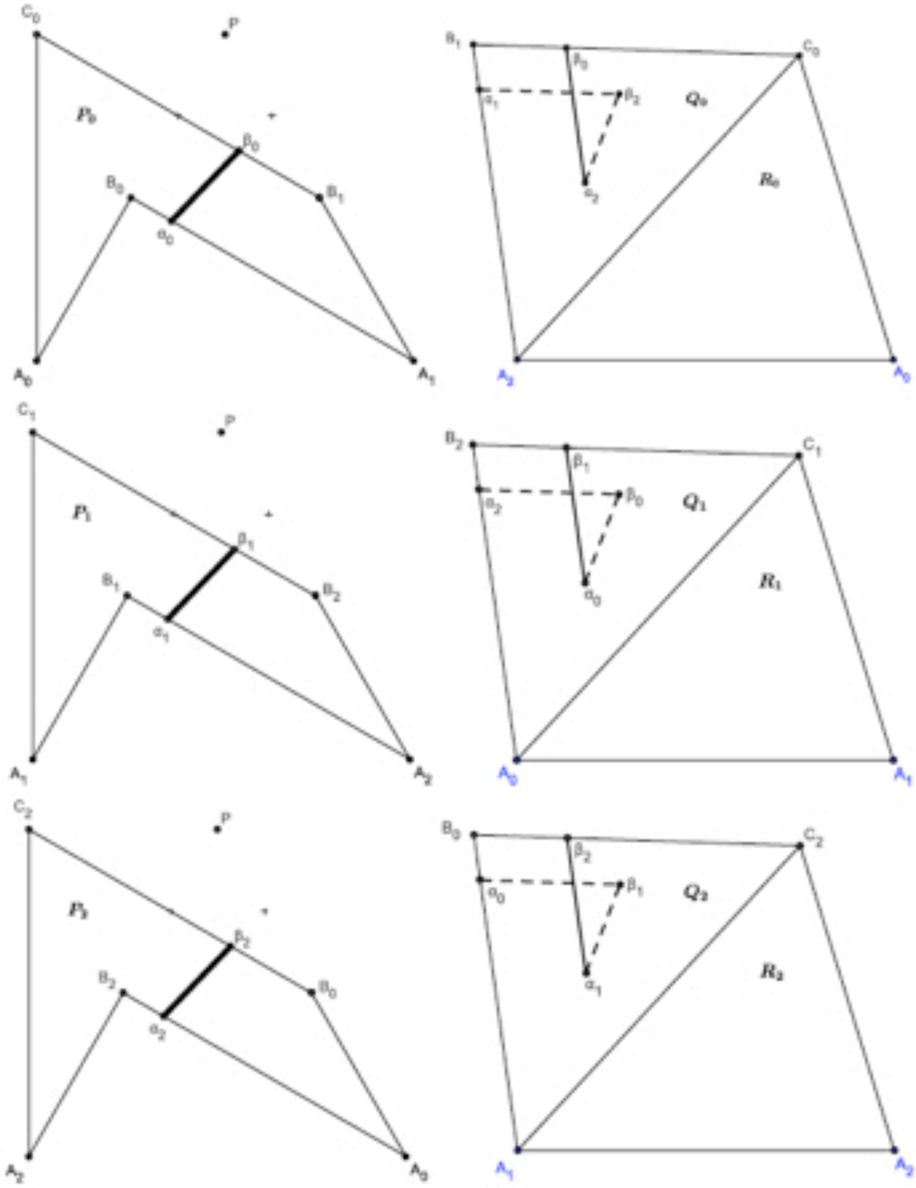

Fig. 3 Assembly of the Boy surface; scale = 0.4. Notation: $\alpha_0 = P_0 \cap Q_2 \cap Q_1$, $\beta_0 = P_0 \cap Q_0 \cap Q_1$. Begin to bring together Q_2 and Q_1, along their intersection line $[\alpha_0\beta_1]$; then insert Q_0 into the previous assembly. The trick is easy and you will succeed quickly. Now add the three pentagons by pushing them through their corresponding slots $[\alpha_0\beta_0]$, $[\alpha_1\beta_1]$ and $[\alpha_2\beta_2]$ on the dorsal faces: the Möbius strip takes its place. Then, to finish add the ventral faces R_0, R_1 and R_2; and if you want to close the model add a last equilateral triangle $A_0A_1A_2$ – which has the same size as the triangle PA_0A_1 – as bottom face. Enjoy! An **important fact** to notice here is that the **flexibility** of the material (paper or rhodoïd) is very useful for the assembly.

A nice description of the construction of the Boy surface and more topological reminders are available in Laura Gay's internship report[7] at the Institute Camille Jordan in Lyon. See also [6] for Boy surfaces having a higher level of symmetry.

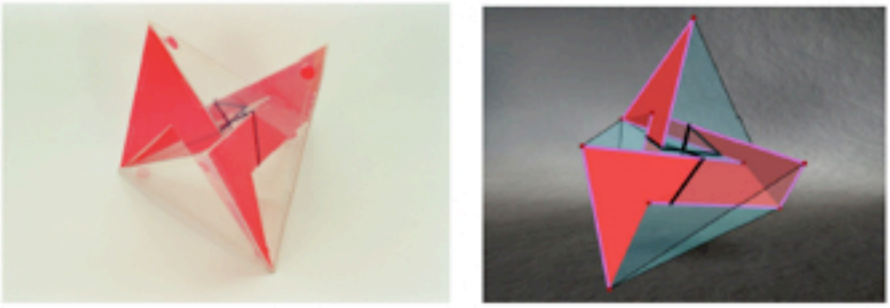

Fig. 4 Minimal Boy surface with 9 vertices: handmade model and JavaView applet.

3 Open halfway-model

With 4 concave pentagons in vertical position we get a model with 12 vertices:

$A_0(3;-3;0)$	$A_1(3;3;0)$	$A_2(-3;3;0)$	$A_3(-3;-3;0)$
$B_0(3;-3;6)$	$B_1(3;3;6)$	$B_2(-3;3;6)$	$B_3(-3;-3;6)$
$C_0(3;-15;8))$	$C_1(15;3;8)$	$C_2(-3;15;8)$	$C_3(-15;-3;8)$

Its 12 faces are $P_i = C_i A_i B_i A_{i+1} B_{i+1}$, $Q_i = C_i B_{i+1} A_{i+2}$ and $R_i = C_i A_{i+2} A_i$ where $i \in \mathbb{Z}/4\mathbb{Z}$.

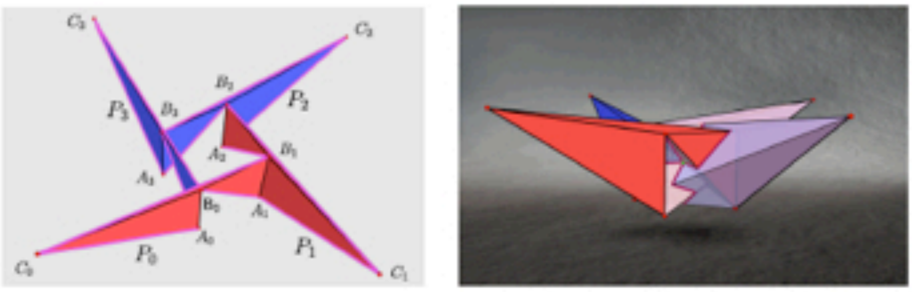

Fig. 5 Open halfway-model: the quadruple point is reachable by passing under the pentagons.

[7] http://math.univ-lyon1.fr/~borrelli/Jeunes/rapport_de_stage_Laura_Gay.pdf

3.1 Intersection line

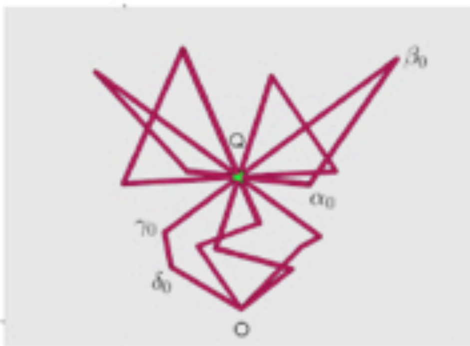

Fig. 6 For the construction of the model – and the JavaView applet – all the coordinates of the points which determine the self-intersection line had to be calculated by solving several linear systems. Here we have annotated $\alpha_0 = P_0 \cap Q_3 \cap Q_1$, $\beta_0 = P_0 \cap Q_0 \cap Q_1$, $\gamma_0 = Q_0 \cap R_0 \cap Q_1$ and $\delta_0 = Q_0 \cap Q_1 \cap R_1$. They all belong to the plane Q_1 like the quadruple point Q (in green).

3.2 Construction of an open halfway-model

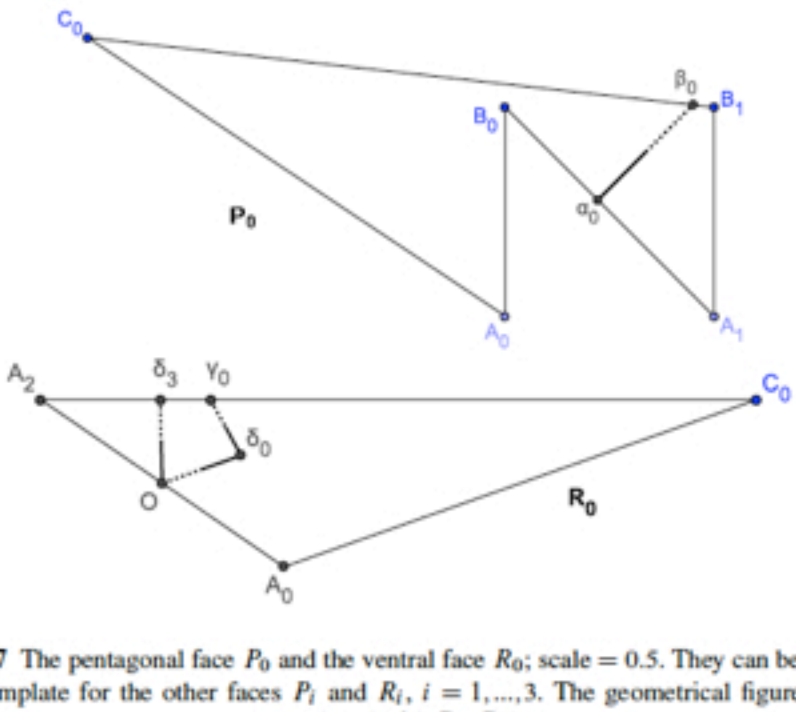

Fig. 7 The pentagonal face P_0 and the ventral face R_0; scale = 0.5. They can be used as template for the other faces P_i and R_i, $i = 1,...,3$. The geometrical figures are reproduced with GeoGebra.

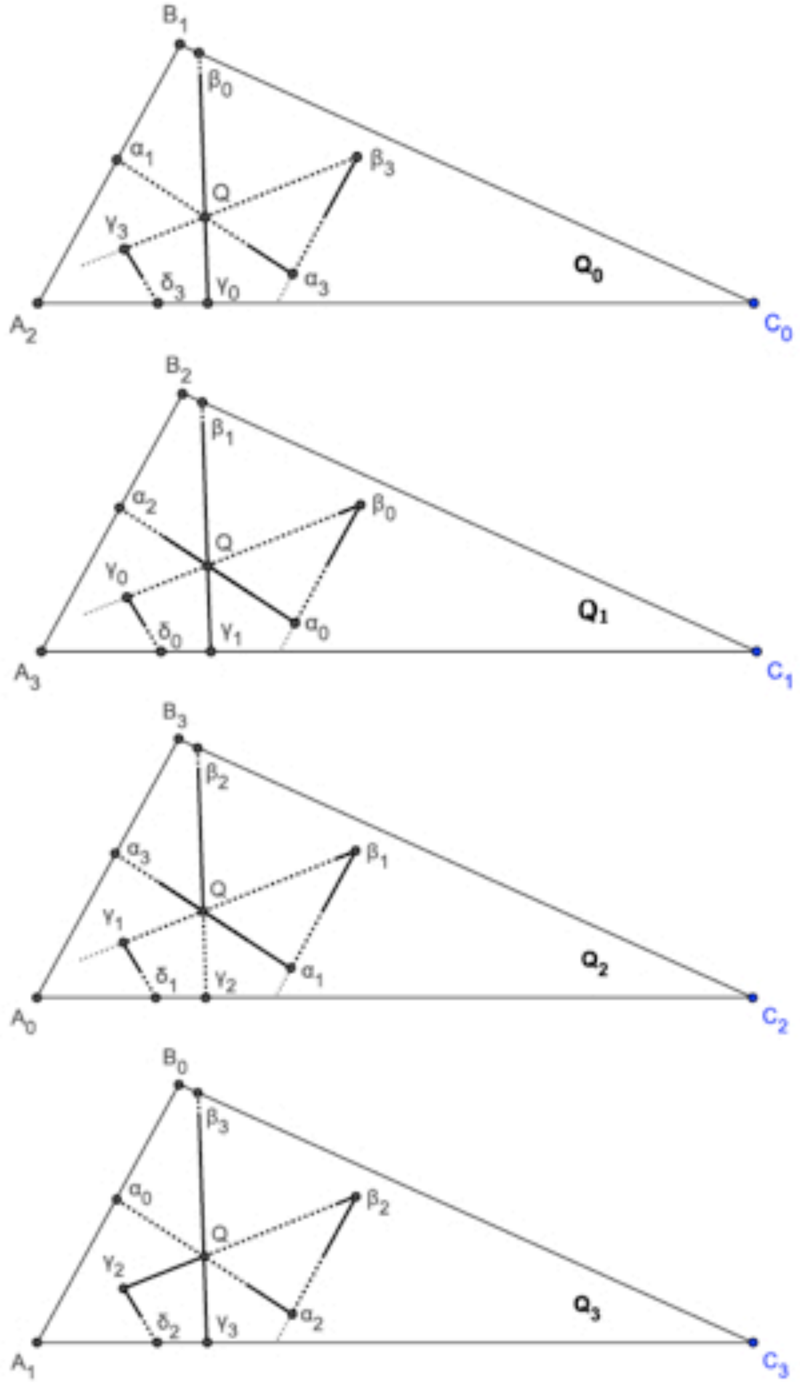

Fig. 8 The four dorsal faces Q_0, Q_1, Q_2 and Q_3; scale = 0.5. They have in common the quadruple point Q. It was crucial to find how to do this assembly.

3.3 Tips for the mounting of the model

Don't use simple paper, it will not work easily. Use at least drawing paper which has a better rigidity.

Fig. 9 Open halfway model: handmade models need ability, precision and perseverance. The pentagons are obtained by gluing together two cardboard sheets – one side in red for the one and one side in blue for the other. To work the rhodoïd, a steel edge and a fine cutter are necessary. To mark the rhodoïd from the plans, needles and a small hammer were used. Faces are fixed together with adhesive tape. The self-intersection line is drawn by using a pencil with permanent ink.

1. Take Q_2 in your left hand.
2. Take Q_1 in your right hand, then push Q_1 into the slot $[\gamma_1\beta_1]$ of Q_2 until the points γ_1 and β_1 of the two faces are touching each other.
3. Now, take Q_0 in your right hand. Try to insert Q_0 into the slot $[\gamma_0\beta_0]$ of Q_1 and at the same time join Q_2 and Q_0 along the segment $[\alpha_1\alpha_3]$.
4. To finish the assembly of the quadruple point, take Q_3 in your right hand. The goal is to push Q_3 through the slot $[\gamma_2\beta_3]$ of Q_0 and at **the same time** Q_1 and Q_3 have to be joined along the segment $[\alpha_0\alpha_2]$. Moreover, Q_2 and Q_3 have to be joined along the segment $[\gamma_2\beta_2]$!
   There is a trick to do this! The flexibility of the matter here is absolutely necessary.
   The trick consists with your left hand to flatten together Q_2 and Q_1 between your thumb and your index finger – level with point β_2 – so that they can be pushed together into the slot $[\alpha_0 Q]$ of Q_3 until they reach the quadruple point Q on Q_3. Then β_2 on Q_2 can move towards β_2 on Q_3. α_2 on Q_1 can move towards α_2 on Q_3. β_3 on Q_3 can move towards β_3

on Q_0. The quadruple point Q can now be assembled by pushing all the points in their right position.
5. Add the four pentagons.
6. Add the four ventral faces.

4 Closed halfway-model of the eversion of the cuboctahedron

This model is better suited to realize the eversion than the previous model.

4.1 Description of the construction

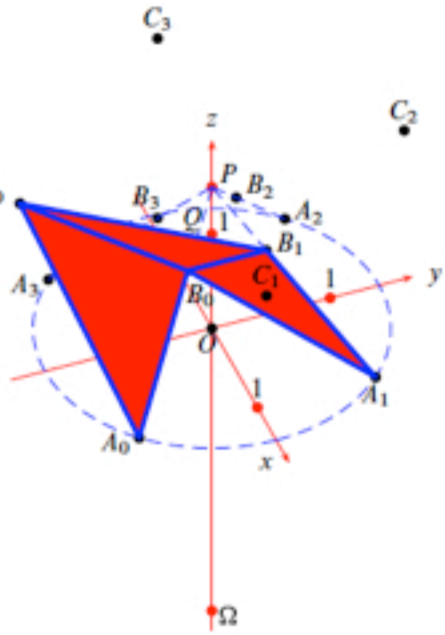

Fig. 10 The pentagon P_0 at the central stage; remember that $Q_0 = C_0B_1A_2$ and $R_0 = C_0A_2A_0$.

On this third model,

1. The four pentagons lean against the lateral faces of the regular pyramid $PA_0A_1A_2A_3$ where the basis is determined by the vertices $A_0(1;-1;0)$, $A_1(1;1;0)$, $A_2(1;-1;0)$ and $A_3(-1;-1;0)$ and where the apex is $P(0;0;\frac{3}{2})$.
2. $B_i \in [PA_i]$ and their third coordinate is 1; furthermore $\boxed{B_i \in Q_{i+1}}$ for $i \in \mathbb{Z}/4\mathbb{Z}$. Consequently, the accesses to the quadruple point Q are closed by the dorsal faces: the halfway-model is said "closed".

3. Let Ω be the point $\Omega(0;0;-3)$ and V_3 the plane $(A_3\Omega A_0)$. Then the coordinates of the vertex C_0 result from $C_0 = P_0 \cap (A_2B_3B_1) \cap V_3$.

All the coordinates can then be calculated. The quadruple point is the point $Q(0;0;1)$.

4.2 Coordinates

$A_0(1;-1;0)$	$A_1(1;1;0)$	$A_2(-1;1;0)$	$A_3(-1;-1;0)$
$B_0(\frac{1}{3};-\frac{1}{3};1)$	$B_1(\frac{1}{3};\frac{1}{3};1)$	$B_2(-\frac{1}{3};\frac{1}{3};1)$	$B_3(-\frac{1}{3};-\frac{1}{3};1)$
$C_0(-\frac{1}{7};-\frac{11}{7};\frac{12}{7})$	$C_1(\frac{11}{7};-\frac{1}{7};\frac{12}{7})$	$C_2(\frac{1}{7};\frac{11}{7};\frac{12}{7})$	$C_3(-\frac{11}{7};\frac{1}{7};\frac{12}{7})$

4.3 Decomposition in several geometries with JavaView

Inside a JavaView applet, it is possible to create different *geometries*; the following pictures illustrate this possibility. Pentagonal faces, dorsal faces, and ventral faces are represented separately for a better understanding of the model. The self-intersection line has also been added after calculation of all the vertices. We touch here the limits of the software: it could be very useful to have a software which allows to isolate directly the self-intersection line, specially

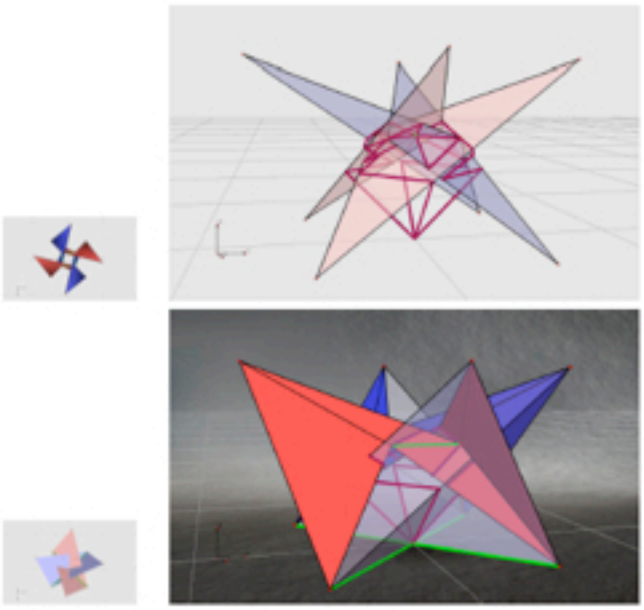

Fig. 11 Closed halfway-model; decomposition in different geometries with JavaView. It is useful to locate the two perpendicular edges $[A_0A_2]$ and$[A_1A_3]$ and the square $B_0B_1B_2B_3$ (in green).

for the study of its evolution along the eversion. This stays actually out of reach with JavaView. The next picture shows the same handmade model photographed in artificial light; the internal subdivision is completely observable. Just under the quadruple point there is a chamber – completely closed towards the outside – which has the shape of an octahedron with four ex-growths like four small teeth. It will be interesting to follow its evolution during the eversion. The second image represents this internal room with the self-intersection line and the quadruple point.

Fig. 12 Closed halfway-model of the eversion of the cuboctahedron. Imagined by Bernard Morin, this model is really the cornerstone of this study.

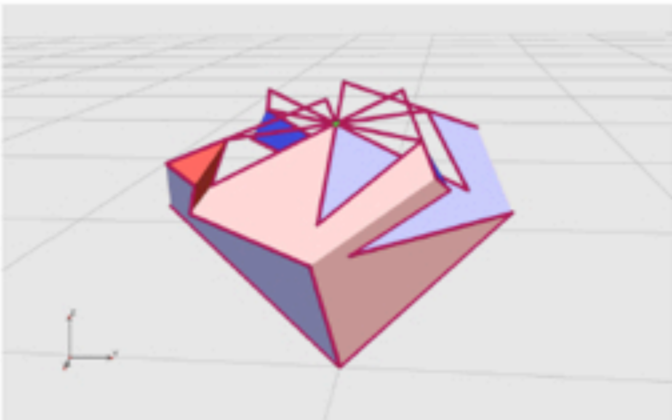

Fig. 13 Internal room under the quadruple point and self-intersection line of the closed halfway-model

See also [7] for halfway models with a higher level of symmetry; the article is illustrated with engravings by Patrice Jeener. The display of the open and the closed halfway-models was enhanced after mail exchanges with Jean Constant. He made two artistic pictures[8] with the use of these models. Enjoy!

5 First eversion of the cuboctahedron

The initial and the final stages of the eversion are obtained by splitting its 6 square faces with 2 orthogonal polar-edges $[A_0A_2]$ and $[A_1A_3]$ and with its equator $B_0B_1B_2B_3$ (in green). By doing so, we get a polyhedron which have exactly the same number of vertices (12), edges (30) and faces (20) as on the triangulated halfway-models!

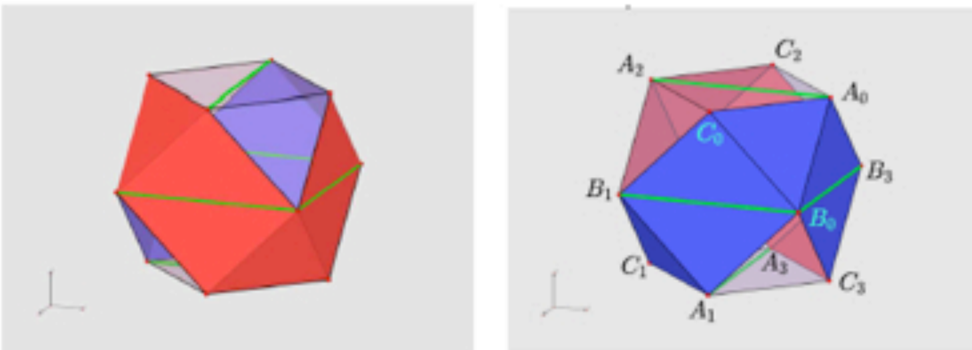

Fig. 14 Initial and final models of the eversion of the cuboctahedron

On the initial model and on the final model, a same vertex has two antipodal positions. Each triangular face is transformed in its antipodal face (see for instance the orientation of the face $A_0C_2B_3$ on the two models), so one can observe that the orientation of the faces has changed on the final model. Similarly, the north polar-edge $[A_1A_3]$ on the first model is changed into the south polar-edge $[A_1A_3]$ on the second model. Observe that these two edges are parallel. The same observation can be done with the south polar-edge $[A_0A_2]$. On the second picture, one can also locate the final position of the pentagon $P_0 = C_oA_0B_0A_1B_1$. Now, the four pentagons of the halfway models are represented by the oscillating belt – composed with 12 triangles – around the equator! The next picture illustrates the problem of the eversion of the cuboctahedron and suggests the question: how does it work?

Bernard Morin conceived a step-by-step deformation which deforms the halfway model by means of *elementary transformations* consisting in moving

[8]http://imaginary.org/fr/node/263

Fig. 15 Initial, halfway and final models of the eversion of the cuboctahedron

a vertex along an edge of the polyhedron. In all, 22 steps are necessary **to transform the halfway model** (step 0) into the final cuboctahedron (in blue). But, **only** 6 steps are needed to obtain a model without self-intersection line! All the models which intervene have a twofold symmetry. What can be done to get the blue cuboctahedron (final step $+22$) from the halfway model can also be done to get the red cuboctahedron (initial step -22). So, if we consider all the 45 models from the model -22 to the model $+22$ then we have all the steps of the eversion!

Fig. 16 First cuboctahedral eversion (Maubeuge 2000). On the picture: Philippe Charbonneau.

A first description of this eversion with annotated pictures is available in my article "Versions polyédriques du retournement de la sphère"[9], Retournement du cuboctaèdre[10] I wrote for the revue *L'Ouvert* [8] of the IREM of Strasbourg. In its "Retournement du cuboctaèdre" [5] François Apéry describes an other eversion which simplifies the previous one with the help of *linear interpolations*.

6 Conclusion

Three models with increasing complexity mark out the way towards the central stage of the eversion of the cuboctahedron. The halfway model represents an ideal point to start the study of the eversion. By building some models, the reader gives himself means to understand better what occurs during a sphere eversion. Animations with JavaView were also realized. Three of them were presented at the conference. This article reminds the long way of maturation and perseverance which preceded their achievement. It is also an encouragement to all those who think that they don't understand maths to believe in their own capacities and to develop them!

Acknowledgments

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References

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[2] B. Morin et J.-P. Petit, "Le retournement de la sphère", *Pour la Science*, 15, (1979).

[3] F. Apéry, "Constructing Wire Models, Mathematics and Art, Mathematical Visualization" in *Art and Education* (C.P. Bruter ed.), Springer-Verlag, 2002, 179-200.

[4] F. Apéry, *Models of the Real Projective Plane: computer graphics of Steiner and Boy surfaces*, Wiesbaden: Vieweg, (1987).

[9] http://mathinfo.unistra.fr/fileadmin/upload/IREM/Publications/L_Ouvert/n094/o_94_32-45.pdf

[10] http://mathinfo.unistra.fr/fileadmin/upload/IREM/Publications/L_Ouvert/n095/o_95_15-36.pdf