Angels and Devils on Triply Periodic Polyhedra

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Abstract

M.C. Escher created a version of his "Angels and Devils" pattern in each of the three classical geometries. In this paper we extend this idea to patterns on triply periodic polyhedra, thus filling combinatorial gaps in Escher's work.

1. Introduction

In this paper we show new "Angels and Devils" patterns on triply periodic polyhedra that were inspired by related patterns of the Dutch artist M.C. Escher. Triply periodic polyhedra have translation symmetries in three independent directions in Euclidean 3-space. Figures 1 and 2 show finite pieces of two such polyhedra decorated with angels and devils. Each of the polyhedra we discuss is composed of copies of a regular

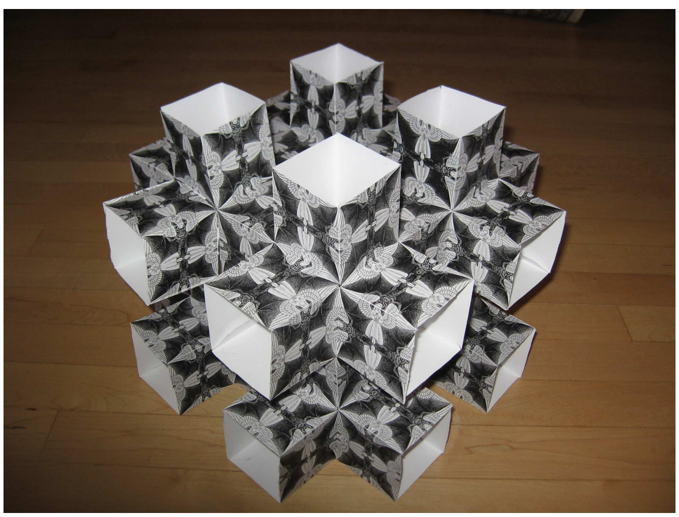

Figure 1: Angels and devils on a piece of the $\{4, 6\}$ polyhedron.

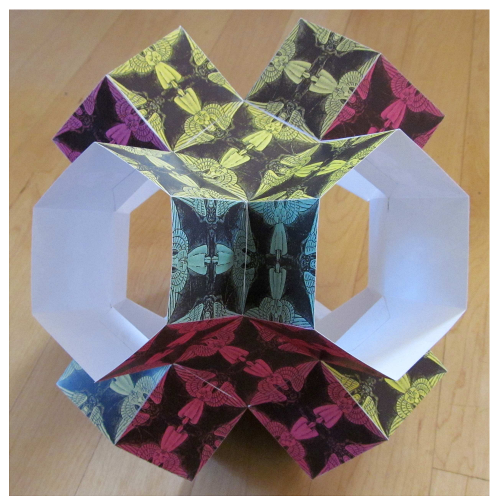

Figure 2: Angels and devils on a piece of a $\{4, 5\}$ polyhedron.

polygon, with more of them around each vertex than would be possible in the Euclidean plane, so we consider them to be hyperbolic. These polyhedra thus have negative curvature, and are related to regular tessellations of the hyperbolic plane. Similarly, the *patterns* we place on these polyhedra are related to patterns of the hyperbolic plane that are based on the corresponding tessellations.

We first review regular hyperbolic tessellations and triply periodic polyhedra, and the relation between them, which extends to patterns on the respective surfaces. Then we analyze Angels and Devils patterns on two polyhedra.

2. Regular Tessellations and Triply Periodic Polyhedra

We use the Schläfli symbol $\{p, q\}$ to denote the regular tessellation formed by regular p-sided polygons or *p-gons* with q of them meeting at each vertex. If $(p-2)(q-2) > 4$, $\{p, q\}$ is a tessellation of the hyperbolic plane (otherwise it is Euclidean or spherical). Figure 3 shows the tessellation $\{4, 5\}$ superimposed on a pattern of angels and devils in the Poincaré disk model of hyperbolic geometry.

We will be interested in infinite, connected *semiregular triply periodic polyhedra*. Such a polyhedron has a p-gon for each of its faces, q p-gons around each vertex, translation symmetries in three independent directions, and symmetry group that is transitive on vertices — i.e. it is *uniform*. We extend the Schläfli symbol $\{p, q\}$ to include these polyhedra (however different polyhdera can have the same $\{p, q\}$). Figures 1 and 2 show pieces of $\{4, 6\}$ and $\{4, 5\}$ polyhedra.

There is often a surface that is intermediate between triply periodic polyhedra $\{p, q\}$ and the corresponding regular tessellations $\{p, q\}$. First, these periodic polyhedra are approximations to triply periodic minimal surfaces (TPMS). Figure 4 shows a piece of Schoen's I-WP TPMS that corresponds to the $\{4, 5\}$ polyhedron of Figure 2 [Schoen]. Second, each smooth surface has a *universal covering surface*: a simply connected surface (the sphere, Euclidean plane, or hyperbolic plane) with a covering map onto the original surface. Since each TPMS has negative curvature (except for possible isolated points), its universal covering surface does too, and thus has the same large-scale geometry as the hyperbolic plane. In the same vein, we might call a hyperbolic pattern based on the tessellation $\{p, q\}$ the "universal covering pattern" for the related pattern on the polyhedron $\{p, q\}$. The pattern of Figure 3 is the universal covering pattern for Figures 2 and 6.

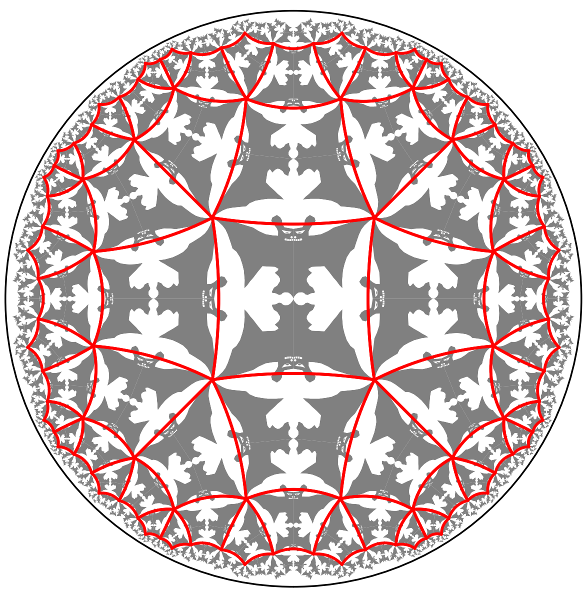

Figure 3: The $\{4, 5\}$ tessellation superimposed on a pattern of angels and devils.

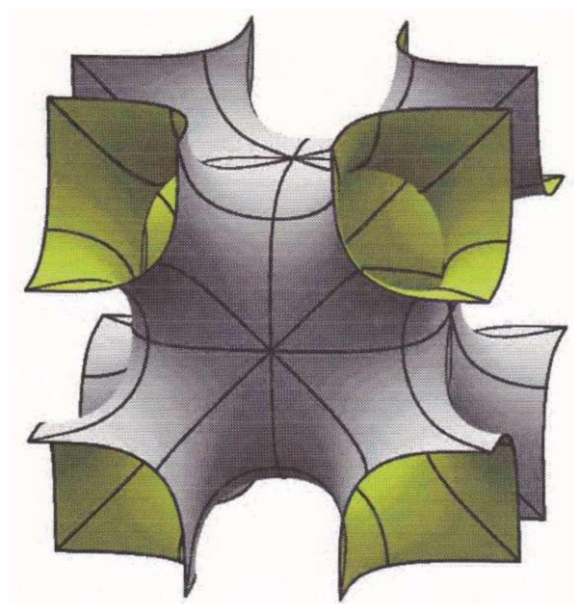

Figure 4: A piece of Schoen's I-WP TPMS which corresponds to the $\{4, 5\}$ polyhedra.

3. Angels and Devils on the $\{4, 5\}$ and $\{4, 6\}$ Polyhedra

Figure 1 shows the $\{4, 6\}$ polyhedron, the simplest triply periodic polyhedron, which is based on the tessellation of 3-space by cubes. The solid within the $\{4, 6\}$ consists of invisible "hub" cubes that are connected by (visible) “strut” cubes, each hub having a strut on each face, and each strut connecting two hubs. The Schwarz P-surface is the corresponding TPMS — it is basically a smoothed out version of the $\{4, 6\}$ polyhedron [Schoen].

Figure 2 shows a piece of a $\{4, 5\}$ polyhedron, which can also be described by the solid within it. That solid consists of truncated octahedral hubs (the square faces of which are visible) with their hexagonal faces connected by regular hexahedral prisms as struts. Figure 2 shows one hub and its 8 connecting struts. As mentioned above, Schoen’s I-WP surface is the corresponding TPMS. Figure 5 shows a $\{4, 5\}$ polyhedron that is actually the same polyhedral surface as that of Figure 1 [Dutch]; Figure 1 shows the outside and Figure 5 shows its “complement”, the inside. The Figure 5 polyhedron is made up of cross-shaped units (shown in different colors), each of which is a cube with four equilateral triangular prisms on it. Figure 6 shows an angels and devils pattern on that polyhedron.

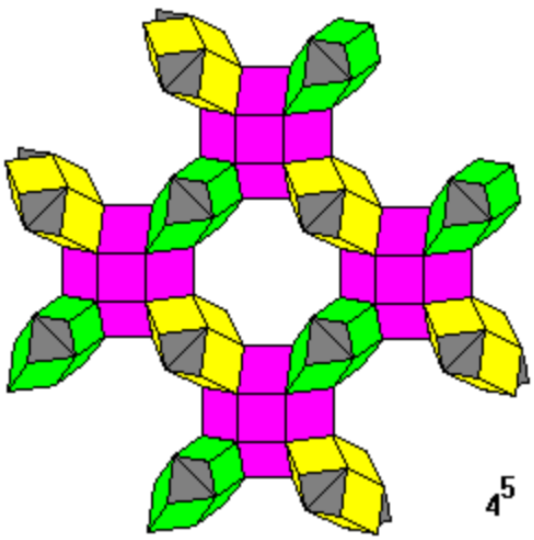

Figure 5: A piece of the $\{4, 5\}$ polyhedron that is the “complement” polyhedron to that of Figure 2.

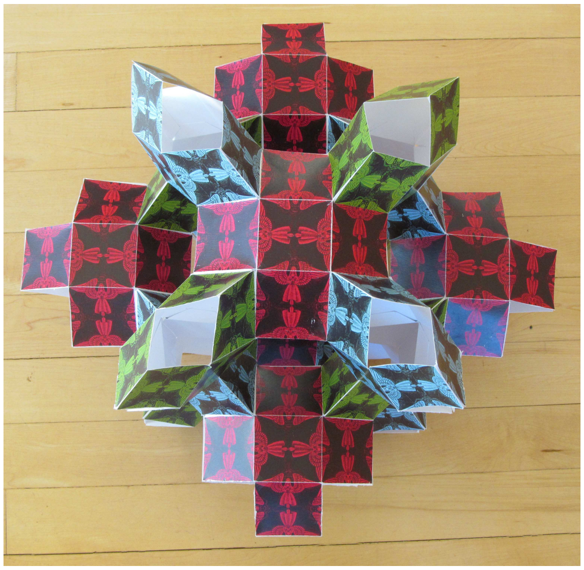

Figure 6: A piece of the $\{4, 5\}$ “complement” polyhedron decorated with angels and devils.

Escher’s “Angels and Devils” pattern, the only one he realized in each of the three classical geometries, were based on the $\{4, 3\}$ (spherical), $\{4, 4\}$ (Euclidean plane), and $\{6, 4\}$ (hyperbolic) tessellations. The patterned $\{4, 5\}$ polyhedra of Figures 1 and 6 thus fill a “gap” between Escher’s $\{4, 4\}$ pattern and the patterned $\{4, 6\}$ polyhedron of Figure 1.

References

[Dutch] S. Dutch, *Wells’ “Hyperbolic Tessellations”*,
`http://www.uwgb.edu/dutchs/symmetry/hypwells.htm`

[Schoen] A. H. Schoen,
`http://schoengeometry.com/e_tpms.html`