An Introduction to the Construction of Some Mathematical Objects

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Abstract In order to understand and to reconstruct the shape of many objects of the geometric world, mathematicians have focused their attention on singularities and deformations. The purpose of this article is to present these usual topological concepts and tools to artists being a priori unfamiliar with mathematics, with the hope that new beautiful creations will appear in the artistic world.

1 Generalities

1.1 Introduction

Artists have often used various kinds of objects in their composition. Mostly images of real objects. All had in common some level of style and representation, pushing forth in their mind the process of stylization. Some created shapes of an abstract nature, as in Egyptian friezes, Roman tilings, or Celtic knots. In essence, there have been two recurring threads in the art of decoration, spirals and tilings [1].

Artists did not develop mathematical theories from their constructs except during the Aeschyleus time and the Renaissance. However, we may consider them as precursors of those theories. Nowadays, many artists are familiar with the larger classes of mathematical objects that appear in their compositions. Using the power of their imagination and the tools with which they defined those objects, could contribute to enrich the catalogue of mathematical objects and the content of their work for our contemporaries and future audiences.

The purpose of this presentation is to give a general introduction to various construction principles frequently used by mathematicians and artists as well. It will

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only focus on the first level of those principles to avoid technical difficulties and complex mathematical theories.

However, we hope some artists will find the following information useful for their work. We will address in particular those who, for whatever reason, are not using the full power of the computing environment and professional software. Salvador Dali's pictorial legacy is a very good example of the way mathematical knowledge can be used by an artist to create paintings of indisputable originality. Artworks related to the cubism expression, such as with the imaginative Chagall, are another example of the potential of human creativity.

Our approach will focus on the topological perspective only, as this rather qualitative choice carries an intrinsic insufficient element of precision. Algebraic and analytical approaches avoid this difficulty but need some mathematical knowledge and training. An advanced introduction to some of those useful techniques was published recently in the Mars issue of the Bulletin of the American Mathematical Society [2]. It should be noted that these structural and quantitative techniques have yet to be developed to reach the full potential of the shapes that are suggested by the qualitative approach. Mathematicians could enrich this qualitative approach with quantitative, numerical controls of all the deformations implied in the construction of objects. As to the qualitative approach, the specific mathematical objects studied by George Francis in *A Topological Picturebook* [3] may also be useful to advanced readers.

We presuppose that all readers understand the notion of (topological) dimension[1] of a space. Since this expose addresses artists and not mathematicians, most other terms will not have the same precise semantic of a larger, more global (mathematical) definition. That choice should allow for a more immediate understanding of the narrative.

In a broad sense, potters, sculptors and, in a more subtle way, painters will be encouraged to find that they share with mathematicians involved in geometry the use of similar processes, deformation and attachment being the most common.

As for deformations, we shall divide them into two opposite types: expansion and restriction, each one being in turn divided into two opposite types, singular or regular.

1.2 Shapes as a Restricted Class of Mathematical Objects

In general terms, mathematical objects can be classified as follows: objects that can be visualized and objects that are too abstract and too general to be a priori physically represented, such as categories or functors.

[1] The topological dimension of a point is 0, of a line, 1, of a surface 2, of our usual space 3, of the space-time 4, etc.

In the following, we will focus only on the class of mathematical objects of the first kind.

Definition: Such mathematical objects will be given the shorter and generic name of *shape*.

Mathematical artists use to deal with immersed or embedded 1, 2 or 3-dimensional shapes. Shapes of dimension one are mostly polygons such as triangles, classical curves as parabolas, knots or fractal lines. In dimension 2, the shapes derive mostly from polytopes, tessellated surfaces, minimal surfaces, topological surfaces, algebraic surfaces. They are often used in their strict mathematical definition and representation as in convenient deformations.

Using a general definition of mathematical objects, artists create their own shapes. To draw them, they use first a pencil or a pen or more directly, numerical symbols.

There are two kinds of numbers: static ones, and dynamic ones.

- A static n-dimensional number is a collection $(x_1, \ldots, x_n)$ of n usual real numbers. Indeed, from the dynamic point of view, it represents a translation.
- A simple dynamic n-dimensional number is slightly more general. It represents a dilatation coupled with a rotation in an n-Euclidean space. When $n = 2$, it is simply a Chuquet number also called a complex or a mixed number. One can do standard algebraic geometry with simple dynamic n-dimensional numbers that are adapted to the control of deformations, such as conformal and quasi conformal deformations in particular.

We will address first those who opt to use the pencil and brushes.

1.3 Characteristic Features of a Shape

When one observes an object, the eye follows a trajectory, a path going from some significant part to another significant part of that object. That path has been defined as the "skeleton of the percept". The parts will be named singular.

In other words, one of the most important characteristic of a shape is its set of singularities: intrinsic singularities and singularities seen from the point of view of the observer. Thus, any change on the singularities of the object is of great significance.

It should be noted that in general, the appearance or disappearance of a singularity or its modification has an important impact on the representation of the object since local or even global curvatures can be drastically changed.

Curvatures are also fundamental characteristic features of a shape. Dramatic changes in curvatures happen on singular parts.

These singular parts play an important role in the setting up of a work of art. They have both a strong significance and a semantic significance. The thorn of a rose, the fang of a vampire, the point of a sword, the blade of a knife are typical singular shapes frequently inspiring fear or violence. They represent symbolic tools of protection and attack, of preservation of one's integrity living in a dangerous

Fig. 1 Canadian Eskimo Art

world. Because they are assigned an essential role in the preservation of the self, they hold in our mind an ambiguous status. Acute angles are somehow aggressive. Drawings and paintings with many straight lines and sharp angles carry a connotation of rigidity and coldness. They also have something in common with the outline of a skeleton and define a kind of structural representation of the object they are associated with.

Definition: A *singular part* within an object of standard dimension n is a subobject of dimension k strictly less than n, which may be connected or not.

Such a singular part is characterized by some local properties of extremality of the object, like for instance the top of the head or the tip of a nose (Fig. 1).

According to our definition, a singular part can only be a point in a 1-dimensional object curve, while in a surface, which is a 2-dimensionnal object, a singular part can consist of points and/or of portion of curve.

The neighbourhood of the singular part is of course said to be regular. What makes the difference between a regular domain and a singular one?

Let us move our finger on the sculpture, first on the head from the left to the right: we observe that the finger goes up until it reaches the top of the head, then it goes down. Thus, there is a dramatic change in the move of the finger when it reaches the top of head creating a singularity: it was going up, now it goes down. If we draw a tangent to the trajectory followed by the finger, the slope of this line is "up" (positive) when the finger is on the left, is "down" (negative) when the finger is on the right (Fig. 2).

Fig. 2 Going up and down

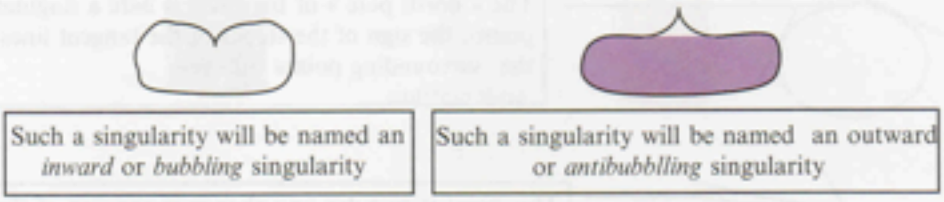

Fig. 3 Bubbling and antibubbling singularities

It is this kind of general phenomenon which is used to characterize a singular part: "dramatic", "catastrophic", "sudden" changes in the directions of the tangent lines or planes in the immediate surroundings of that singular part.

In a singular point, a discontinuity occurs into the sign or into the value of the slopes of tangents belonging to the outer edge of that point.

As a result, most of these singular parts can be obtained by a pinching process of the set of internal modifications and deformations of the object as described below.

2 Internal Modifications

2.1 Pinching

Definition: We shall call *pinching*, denoted by $P_{n \to k}$, the process of smooth deformation that transforms a regular domain of the shape of dimension n into a domain of a singular part of dimension k less than n.

Indeed, a pinching process on any part of the object is obtained by creating discontinuities into the sign or into the value of the slopes of tangents belonging to the surroundings of that part.

On a curve, pinching occurs at points, while on a surface pinching can occur at points or along curves taking the character of singular parts.

Example 1: Geometrical (Figs. 3 and 4)

Example 2 (Fig. 5): Using antibubbling singularities, and indeed some other topological tools, could an artist have created the following lilac flower? (Fig. 6):

Example 3: An other geometrical object, the Eigthy.

Take a tube, an hollow cylinder. You can pinch it along a generatrix, in a bubbling way, or in a antibubbling way. The original generatrix becomes a singular line. Note the important reversible character of the operation (Fig. 7).

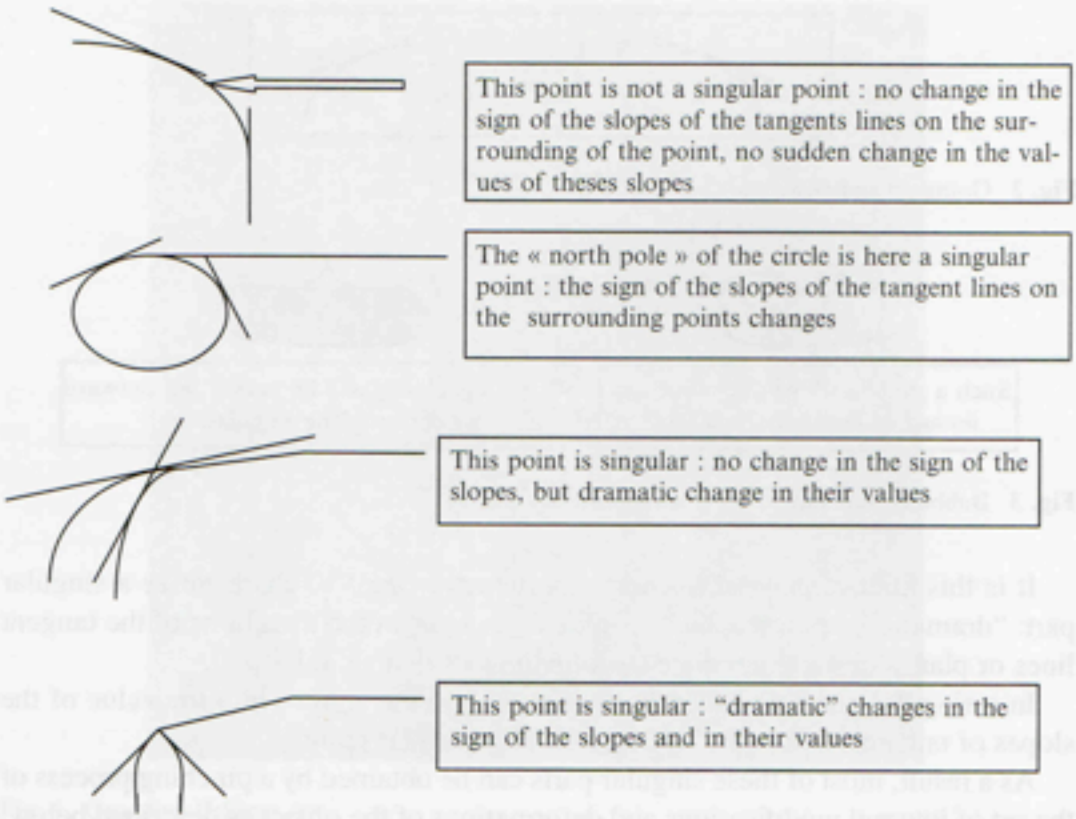

Fig. 4 Regular and singular points

Consider the bubbling cylinder. You can make a second antibubbling pinching in such a way that the two singular generatrix can be confounded. You can even pinch locally the result into a singular point and paint it to obtain the following figure called the Eigthy (Fig. 8):

Other topological techniques allow to construct this figure.

Before leaving the usual notion of singularity, let us observe another kind of phenomenon. Suppose that some part of a curve begins to vibrate more and more strongly. It may break into very small pieces, getting smaller and smaller until it finally dissolves into points, making a continuous, quasi continuous or even a discrete set. We would call this phenomenon a *fractal singularisation*, and its result a *fractal singularity* (Figs. 9 and 10).

2.2 *Inflations*

There are two kinds of inflations, singular inflations and regular inflations.

The singular inflations are the most interesting: they are attached to metamorphosis and the creation of new unexpected shapes and behaviours.

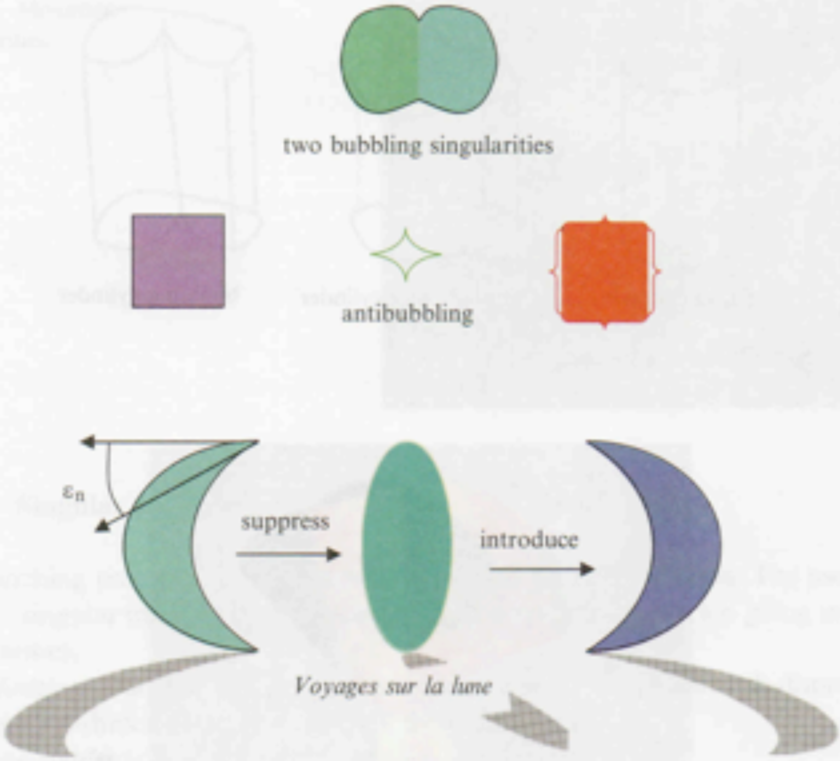

Fig. 5 Examples of births and evolutions of singularities

Fig. 6 Orchid: *Paphiopedium venustum*. On the *left* and *right* of the flower, two antibubbling singularities

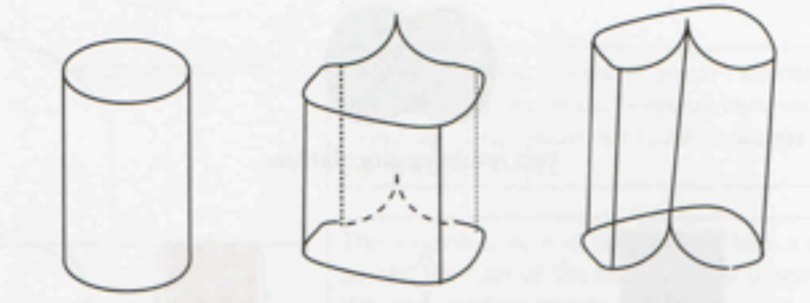

Fig. 7 Singular lines on a surface

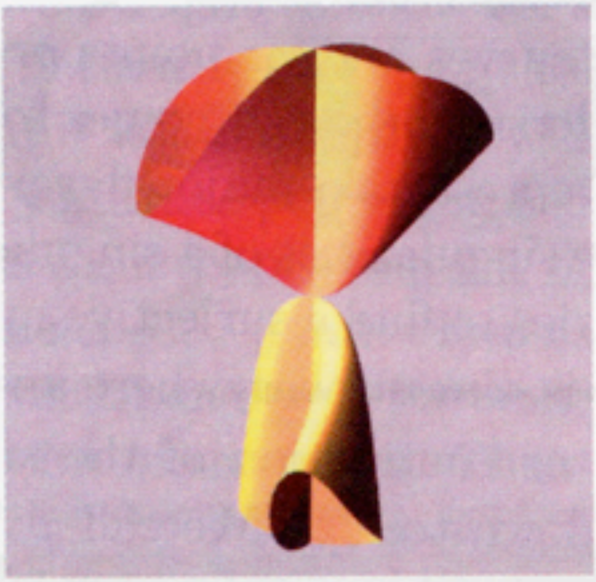

Fig. 8 The Faber-Hauser Eigthy

Fig. 9 Fractal singularities

Regular inflation are used in art to trigger emotion on the mind of the observer and emphasize didactic messages: El Greco (in most of his works) stretches his characters' features to express the yearning of the soul for God. Honoré Daumiers' caricatures and Hieronymus Bosch's figures, also represent two kinds of inflation.

Fig. 10 Mountain singularities

2.2.1 Singular Inflations

The pinching process has a converse we will call singular inflation. The use of the words "singular inflation" is restricted here to singularities. We are going to inflate singularities.

Definition: Denoted by $I_{k \to p}$, a *singular inflation* transforms a k-dimensional part into a p-dimensional part ($k < p$), the inflated part.

Done suddenly, it will be here called a *blowing-up*.[2]

The inflated part is related to its singular generative part by a few properties among which a trivial but fundamental property: the singular generative part can be obtained from the inflated part by a continuous deformation leaving invariant the topological properties of the successive deformed parts, properties having an exceptional character for the singular part.

Note: A pinching process compresses a part of dimension n into a part of lower dimension k. Usually, there might be many acceptable such parts of lower dimension, even when we restrict k to $n-1$. Additional constraints can of course reduce the amount of possibilities.

A similar assertion can be done concerning singular inflations: for instance a point can be inflated into a segment, a circle, a sphere, or a 2-disk, a 3-ball, etc. The general procedure to inflate an object is of course to go step by step and increase the dimension step by step (Fig. 11).

Example 4: In these two examples symbolized by $I_{0 \to 1}$, a singular point blows up into a circle or in a shape having the same topological properties as the circle (Figs. 12 and 13).

Example 5: Note that in the two preceding cases, the singular point could even blow up into a disk, a two-dimensional object, which in turn can be flat, bubbling, antibubbling, and mixed (Fig. 14).

[2] "Blowing up" is a standard mathematical term (in French "éclatement"). Mathematicians call the inflated part the "blowup" of the singular part.

Fig. 11 Singular inflations into circles of various sizes of one point

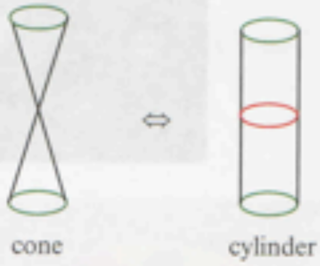

Fig. 12 A singular point inflates into a circle

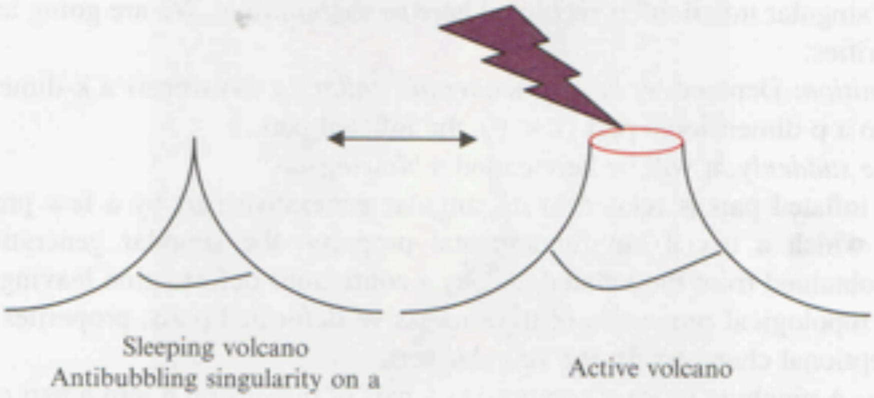

Fig. 13 A singular point inflates into a circle

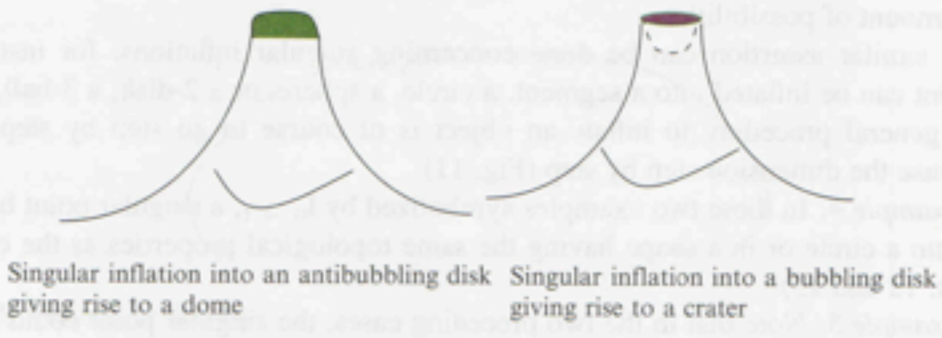

Fig. 14 Some singular inflations into a two-disk of the singular point or of the circle of the Example 4

Note that when the 2-disk is not flat, it can be deformed into a tube open on one side, whose axis can be any non closed curve. The classification of knots can be used to classify these curves.

Fig. 15 The dome of Fig. 14, left, can be deformed into these topologically equivalent tubes

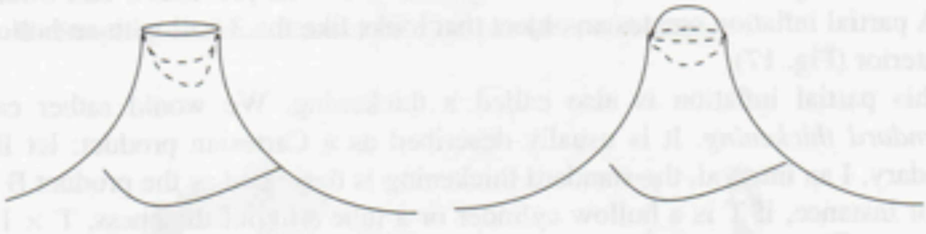

Fig. 16 Singular inflations of a singular point with multiplicity 2, left two bubbling, right one bubbling and one antibubbling

The artist will enjoy the drawing of such a tube nicely winding from and around the surface and the circle on which it arises. From that tube, fanciful horns are sometimes growing up or vanishing, throwing fantastic beams of light which illuminate an unexpected choreography.

The tube may simply show an undulation, that disappears when the viewer moves away from the initial singularity (Fig. 15):

Example 6: In the previous example the singular point blows up into a single 2-disk. But we may also consider the possibility that this point blows up into *a multiplicity* of disks having the same circle as boundary. There are particular cases of such an occurrence. We will select the simplest one, when the multiplicity is two, one disk being bubbling, the other one antibubbling: we get a 2-sphere $\mathbf{S}^2$ since the 2-sphere can be constructed by identifying the circles which borders two disks.

Indeed, the singular point can be blown up into the 2-sphere since, conversely, that sphere can be continuously shrink into a point. Note that again, from our topological point of view, the sphere can be replaced by any shape having the same topological properties (Fig. 16).

If exceptionally there is a continuous infinity of this type of double expansion, the spheres may completely fill a bowl represented here by the symbol $\mathbf{D}^3$.

Example 7: Let's take an orange as the physical representation of such a bowl and the 2-sphere $\mathbf{S}^2$ as its boundary. We may consider that boundary as the singular part of the bowl.

There are two main ways to inflate the 2-sphere, either inwards towards the centre of the bowl, or outwards. Any such singular inflation may be partial or complete. A complete inwards inflation of the 2-sphere is the 3-ball $\mathbf{D}^3$. A complete outwards

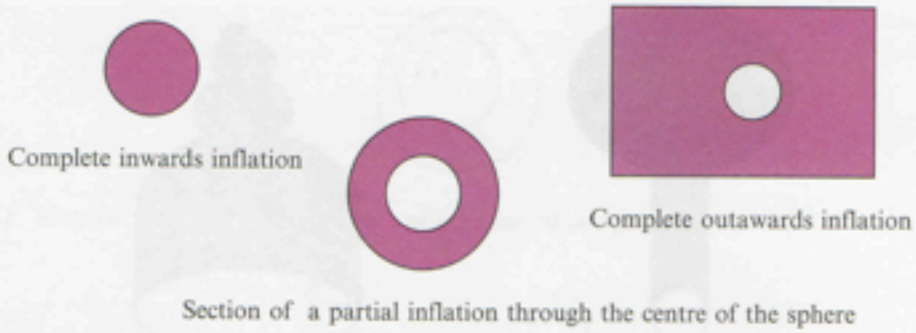

Fig. 17 Some different inflations of the border of the usual 2-sphere

inflation fills up the usual 3-space leaving an hole that the previous 3-ball would fill up. A partial inflation creates an object that looks like the 3-ball with an hollow in its interior (Fig. 17).

This partial inflation is also called a thickening. We would rather call it a *standard thickening*. It is usually described as a Cartesian product: let B the boundary, I an interval, the standard thickening is described as the product B $\times$ I.

For instance, if T is a hollow cylinder or a tube without thickness, T $\times$ I will denote a tube whose local thickness is I; or if $\mathbf{D}^3$ is a 3-usual ball whose border is an usual sphere of radius 1, the standard thickness of $\mathbf{D}^3$ will be a 3-ball whose radius is 1+ the length of the interval I.

More generally, let C be any other object: the Cartesian product B $\times$ C may be understood as a thickening of B through C.

2.2.2 Regular Inflations and Desinflations

These transformations can be global or local. Optical illusions, anamorphoses introduce increasing or decreasing local sizes through lengths and twists. The perspective theory has formalized some of those transformations of size that do not change the topological properties of the shape. Inflation is an important component of visual communication expressing power, will, and hopes.

Foldings are frequently used as a first step in the process of developing transformations.

2.3 *Folding*

Definition: *Folding* is the operation by which one can act on a part of an object and change the local curvature along the points of that part, and eventually the size of that part.

There are two types of foldings: continuous foldings, and singular foldings as in the art of origami—a paper folding methods. The folding of a domain is continuous when the direction of the perpendicular to the tangent line or plane along any

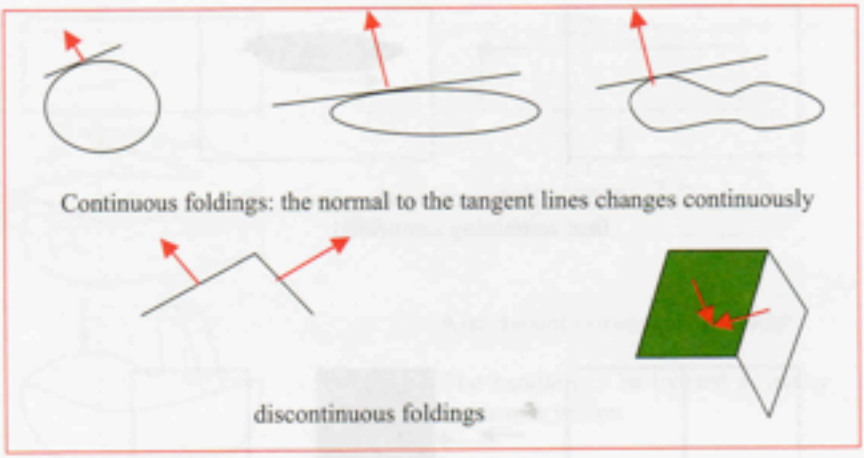

Fig. 18 Simple geometrical illustrations of the two kinds of foldings

Fig. 19 Three knots drawn by George Francis

transversal line to the domain changes continuously. If a discontinuity appears somewhere in that change of direction, the folding is locally singular (Fig. 18).

Example 8: In 2-dimension, when drawing on a sheet of paper, the folding of a line can use not only changes of its length but its rotations as well. In 3-dimension, the folding of a line uses changes of its length and *twists* which are couples of simultaneous rotations in two non parallel planes. Note the semantic and artistic importance of twists, expressing at once force and motion as the works of Michelangelo and El Greco titled *Laocoön*.

Examples of geometrical twists as in the work by George Francis [3] (Fig. 19):

2.4 Cutting and Opening

Definition: *Cutting* is an act of separation, of disconnection, done along any k-dimensional part of an object of dimension n (>k).

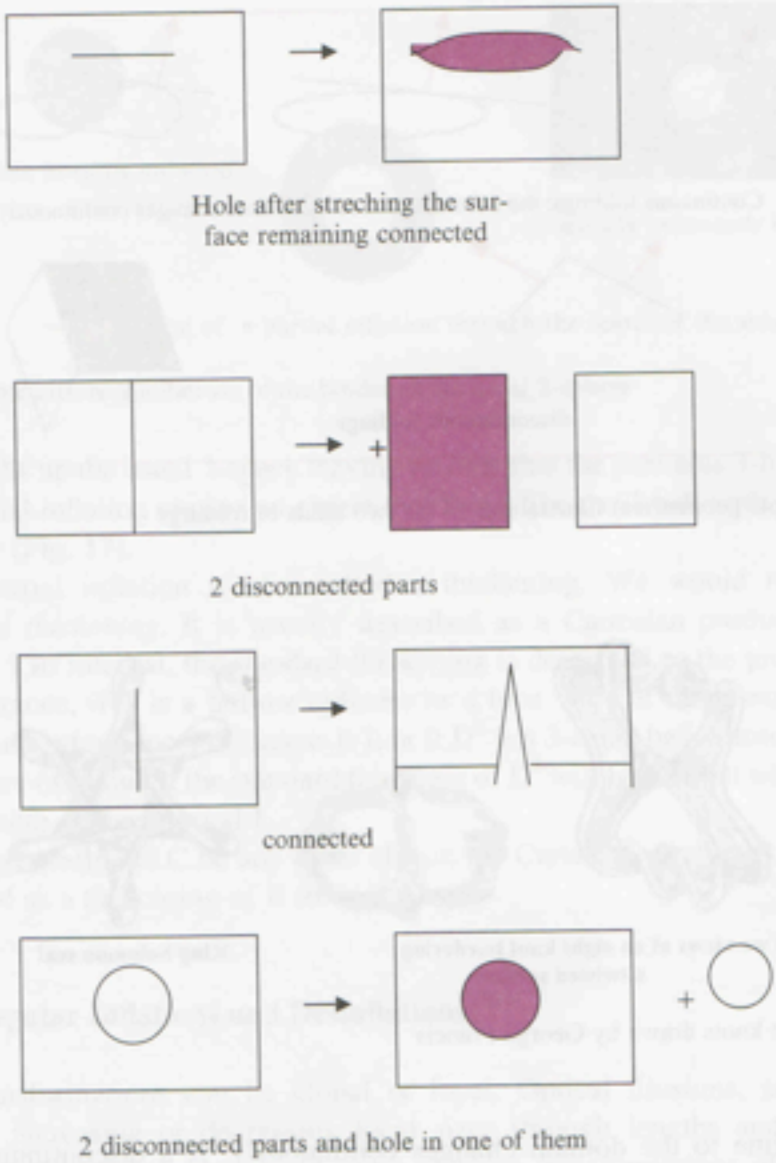

Fig. 20 The four different effects of a cut

Any surface can be incised at any of its points, and cut along any of its curves. It will create two diffeomorphic curves that will be called the *lips* of the cutting. They belong to the boundary of the surface and from this fact are singular parts.

After cutting of a surface along one of its curve, one of the four following situations may happen (Fig. 20):

Such a cut may separate the surface into disconnected pieces. It happens each time the curve is the boundary of one or several disks on the surface (case 2 and case 4 where the surface has a curve as boundary, the curve along which the cutting is done meets that boundary into two points, creating with the boundary at least one loop which is the boundary of a 2-disk).

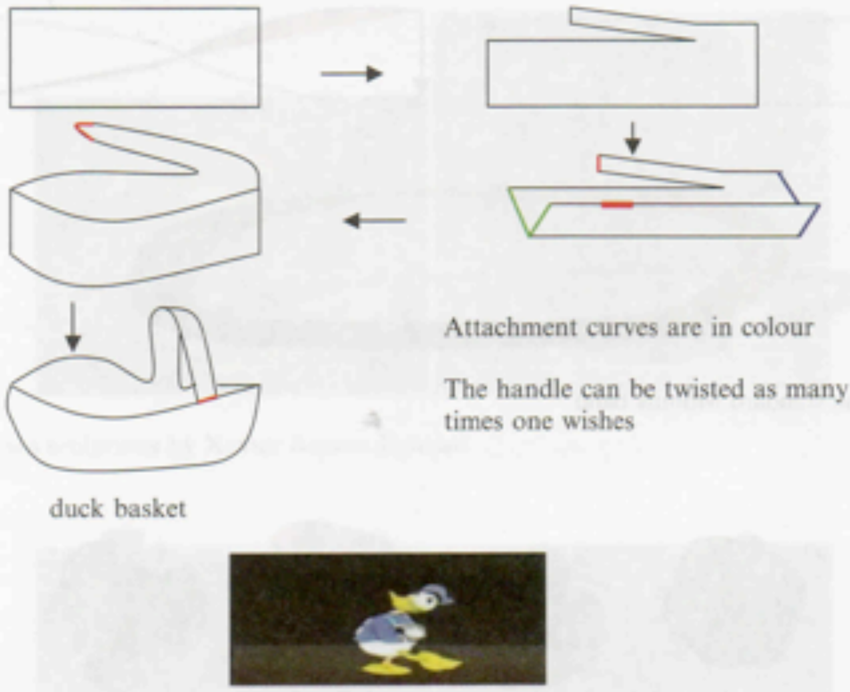

Fig. 21 Duck like Donald

3 External Modifications

The cutting process, which may introduce separation into pieces, makes a transition between internal and external modifications. If a separation has been introduced through cutting, conversely, it is supposed that the inverse operation of joining the two primitive separated pieces is possible. A good glue is all we need.

A *gluing process*, also named an *attachment process*, is considered here as a change on the objects that are attached from the exterior side.

Such an addition is built along *domains of attachment*: it can be a point, a piece of line, a piece of surface. The addition process supposes that the two objects that will be attached share a similar domain which will be used as domain of attachment.

It should be noted that the process of adjunction can also work internally.

A simple example is the creation of a basket from a rectangle, involving cutting, folding, and attachment. There are several ways to build this basket. Here is an example (Fig. 21):

When the two domains of attachment belong to the same object, the attachment process will be here called an *identification*.

The Möbius band is a classical result of an identification process. Take a band of paper whose shape is a rectangle. Orient the two small edges in opposite directions, twist the band an odd number of times, then you can glue (identify) the two small edges since they have now the same orientation (Fig. 22).

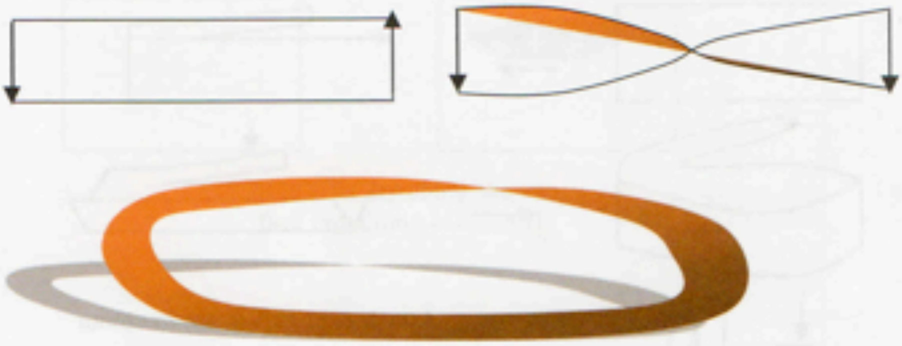

Fig. 22 The standard Möbius band

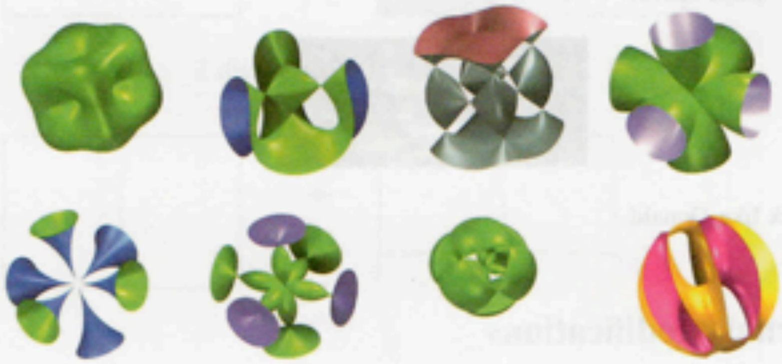

Fig. 23 Jeener's games with surfer

4 Synthesis

All the objects can be constructed using the following operations in the preceding sense: pinching, inflating, folding, cutting, attaching.

Here are some classical mathematical objects that can be built in that ways. They were sent to me by Patrice Jeener and produced with the software "surfer" (Fig. 23).

These objects were created by solving only one polynomial equation we will write under the abridged form $p(x_m, y_n, z_p) = 0$. Though for one polynomial equations the different classes of singularities are limited, one can expect an infinite number of shapes since the values of the integers m, n and p are themselves infinite: the human imagination cannot a priori reach the totality of the variation and subtle differences that exist between those mathematical shapes, all the more we consider here only one dimensional polynomials, whereas we can for instance consider projections onto 2- or 3-dimensional spaces of objects defined by various types of equations in multidimensional spaces. Except for a very few, like the ball or the cylinder, most of those mathematical objects do not have any significance to us, at this time: they are unfamiliar and considered as artificial; having no meaning, they seem to be cold and lifeless. But we cannot foresee the future. Humanity is evolving. Subjective interpretation may be giving way to more effective rational

Fig. 24 Two sculptures by Xavier Bonnet-Eymard

Fig. 25 Repetition in nature

thinking. Those objects may get a greater interest because they speak to our rationality through an intellectual training that teaches us how to look at them, how to see their properties and qualities. Though they may look richer by increasing their internal symmetries, each of these mathematical objects presented individually carries some level of melancholy due in part to their isolation.

Many artworks do not consist in the presentation of a single object. It happens of course: in that cases, the object has a sufficient strength of expression and richness *in se*, and sometimes appears as a composition of various objects. Sculpture, where the qualities of the material play an important role, is typical from this perspective (Fig. 24).

Artists rather create compositions. Several standard components play a role in those creations such as light, slightly distorted symmetry, abundance (mainly by repetition), perspective (from classical to reverse or frontal as in many Chagall' works) (Fig. 25). All these elements are related to physical fundamental principles and facts.

Maybe children, young and old ("Heraclitus called children' games men' thoughts"), will enjoy playing with some mathematical objects such as the ones we discussed before. They will be able to build friezes and free standing objects, fill their space with new creations, cut new shapes, create and hold new flowers, make new connections "à la Chagall" by inserting various objects and material in new composition. In that way, mathematics will be, as before, at the service of art.

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